TOPOLOGICAL ASYMPTOTIC ANALYSIS FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

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Abstract

Topological asymptotic expansions for quasilinear elliptic equations have not been studied yet. Such questions arise from the need to apply topological asymptotic methods in shape optimization to nonlinear elasticity equations as in imaging to detect sets with codimensions ≥ 2 (e.g. points in 2D or segments in 3D). Our main contribution is to provide topological asymptotic expansions for a class of quasilinear elliptic equations, perturbed in non-empty subdomains. The obtained topological gradient can be split into a classical linear term and a new term which accounts for the non linearity of the equation. With respect to topological asymptotic analysis, moving from linear equations onto nonlinear ones requires to heavily revise the implemented methods and tools. By comparison with the steps carried out to obtain such expansions with the Laplace equation, the core issue for a quasilinear equation lies in the ability to define the variation of the direct state at scale 1 in $\mathbb{R}^N$. Accordingly we build dedicated weighted quotient Sobolev spaces, which semi-norms encompass both the L^p norm and the L^2 norm of the gradient in $\mathbb{R}^N$. Then we consider an appropriate class of quasilinear elliptic equations, to ensure that the problem defining the direct state at scale 1 enjoys a combined p and 2 ellipticity property. The needed asymptotic behavior of the solution of the nonlinear interface problem in $\mathbb{R}^N$ is then proven. An appropriate duality scheme is set up between the direct and adjoint states at each stage of approximation.

Keywords: *quasilinear elliptic equations, topological asymptotic analysis, topological derivative, two-norm discrepancy, quasilinear interface problems.*

1. Introduction and overview

The present article provides topological asymptotic expansions for a class of quasilinear elliptic equations of second order.

The methods of so-called *topological asymptotic expansions* or *topological gradients* or *topological sensitivity* have been developed since the 1990's [33, 47, 51, 58, 59]. They are applied in the field of shape optimization (e.g. [4, 14, 33]) as well as in image processing (e.g. [16, 19, 20, 21, 22, 23, 24, 25, 41, 42]). The key idea is to assess the sensitivity of an appropriately chosen functional, taken on the solution of a partial differential equation, when the latter is perturbed in the vicinity of a given point x_0, in a subdomain of which one geometric parameter goes down to zero.

More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and consider a partial differential equation in Ω, e.g. the Laplace equation, with a boundary condition on boundary $\partial\Omega$. Assume this equation admits a unique solution u_0, called the *unperturbed direct state*, in an appropriate functional space $\mathcal{F}_0$. Consider a bounded domain $\omega \subset \mathbb{R}^N$, such that $0 \in \omega$, a point $x_0 \in \Omega$ and a parameter $\varepsilon > 0$ small enough such that $\omega_\varepsilon := x_0 + \varepsilon\, \omega \subset \Omega$. Then, as shown on Figure 1, modify the equation in ω_ε, either by changing a coefficient of the equation in ω_ε, for instance a conductivity, or by restricting the domain of the equation to $\Omega_\varepsilon := \Omega \setminus \bar{\omega}_\varepsilon$ and by requiring an additional boundary condition on $\partial\omega_\varepsilon$. Assume that the perturbed equation obtained that way admits a unique solution u_ε, called the *(perturbed) direct state*, in a functional space $\mathcal{F}_\varepsilon$. Let now $J_\varepsilon : \mathcal{F}_\varepsilon \to \mathbb{R}$ be a functional defined for $\varepsilon \geq 0$ small enough. Then the *topological asymptotic expansion* of J_ε is an expansion of the form

$$J_\varepsilon(u_\varepsilon) = J_0(u_0) + \rho(\varepsilon)\ g(x_0) + o(\rho(\varepsilon)), \quad \forall\ \varepsilon \geq 0 \text{ small enough}, \tag{1.1}$$

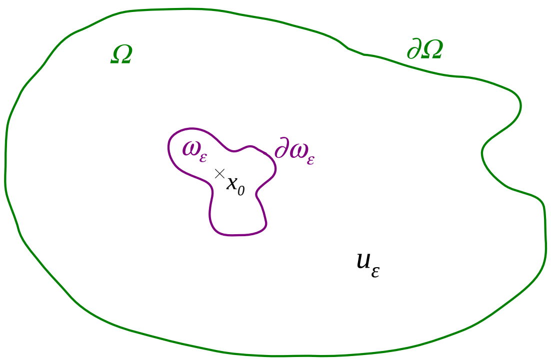

FIGURE 1. An equation perturbed in ω_ε

where ρ is a non negative function such that $\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0$. The scalar $g(x_0)$ is called the *topological gradient*, or *topological derivative*, at point x_0. When x_0 ranges in Ω, one defines that way the *topological gradient* field $g : \Omega \to \mathbb{R}$.

Topological asymptotic expansions were obtained for many equations such as linear elasticity equations [33], Helmholtz equation [57], Stokes equations [36] and incompressible Navier-Stokes equations [11]. Regarding the Laplace equation, topological asymptotic expansions were obtained in the case of Dirichlet boundary condition [12, 35] as in the case of Neumann boundary condition [15, 52]. Moreover asymptotic expansions were provided for the first eigenvalues and eigenfunctions of classical problems for the Laplace operator perturbed in small domain ω_ε in 2 and 3 dimensional domains [50, 51].

Hence from a mathematical perspective, the question of topological asymptotic expansions for nonlinear elliptic equations of second order naturally arises next. The case of semilinear equations, made of the Laplace operator added to a nonlinear term, was studied in [12, 38].

- The topological sensitivity for a class of nonlinear equations of the form
$$-\tilde{\Delta}u + \Phi(u) = \sigma, \quad u \in H_0^1(\Omega)$$
where $-\tilde{\Delta}$ is a linear and homogeneous differential operator of order 2 and Φ is a possibly nonlinear function, is studied in [12]. The functional setting is that of Hilbert spaces.
- Topological derivatives for equations of the form
$$-\Delta u = F(x, u(x))$$
where Δ is the Laplacian operator and $F \in C^{0,\alpha}(\Omega \times \mathbb{R})$, are obtained in [38]. The functional setting is that of weighted Hölder spaces.

To our best knowledge, topological asymptotic expansions remain unknown for nonlinear elliptic equations with a nonlinearity in the principal part of the differential operator, such as quasilinear equations.

Moreover such questions also arise from at least two applicative fields.

(1) In the field of shape optimization, the use of linear elasticity equations remains a drawback whenever the actual behavior of mechanical structures is better described by equations of nonlinear elasticity [30]. This issue was raised e.g. in [3] §8.
(2) In the field of imaging, the detection of subsets of codimension ≥ 2, as points in $2D$ and curves in $3D$, remains an important task, e.g. in medical imaging. A smooth curve can be locally approximated by a segment, of length 'small enough'. Applying a topological asymptotic method, the task of detecting segments in $2D$ images was

dealt with in [10] by means of a Laplace equation with Neumann boundary condition. According to the theory of potential [2, 37, 46, 49], the Laplace equation can only detect subsets which codimensions are < 2. For instance it cannot detect points in $2D$ or segments in $3D$. For such tasks, one may consider the p-Laplace equation, where parameter p is chosen strictly larger that the codimension of the subsets to detect.

In accordance with such motivations, the present article provides topological asymptotic expansions for a class of quasilinear elliptic equations of second order, perturbed in non-empty open subsets.

We first analyze in section 2 some of the specific issues arising in the process of obtaining topological asymptotic expansions for quasilinear elliptic equations. To serve as a reference, we sketch the steps usually taken for that purpose in the case of a linear elliptic equation. Then considering the p-Laplace equation, we raise the conditions required to define the variation of the direct state at scale 1 in $\mathbb{R}^N$, denoted H.

These conditions justify that we build dedicated quotient weighted Sobolev spaces in appendix A. We thus introduce Banach spaces denoted $\mathcal{W}(\mathbb{R}^N)$ and $\mathcal{V}(\mathbb{R}^N)$ and the Hilbert space $\mathcal{H}(\mathbb{R}^N)$, all of them enjoying Poincaré inequalities. Eventually Proposition A.5 will be pivotal to ensure a combined p- and 2-coercivity to the nonlinear operator defining the variation of the direct state at scale 1.

The conditions pointed out in section 2 also widely determine the class of quasilinear elliptic equations which we define in section 3. In particular the corresponding operators satisfy a combined p- and 2-ellipticity property. Our main contribution is the topological asymptotic expansion stated in Theorem 3.5. To obtain this result, we study the direct state and its variation at each stage of approximation. We similarly study the variations of the adjoint state. The steps taken for the adjoint state are classical as we define the adjoint state as solution of a linearized equation. By contrast the nonlinear approach applied to the direct state is fairly new. The needed asymptotic behavior of the solution of the nonlinear interface problem in $\mathbb{R}^N$ is then proven. An appropriate duality scheme is set up between direct and adjoint states at each stage of approximation. We eventually prove the topological asymptotic expansion of the functional, separating a linear term and a nonlinear term. While both terms depend on the variations of the direct and adjoint states at scale 1 in $\mathbb{R}^N$, one essential ingredient for the nonlinear term is an operator, denoted S, characterizing the nonlinearity of the considered equation.

For reader's convenience, most proofs are postponed to section 4.

Let us end this introduction by gathering some notation used throughout this article. The space dimension is denoted by N, $N \in \mathbb{N}$, $N \geq 2$. A real number $p \in [2, \infty)$ and its Hölder conjugate exponent q such that $1/p + 1/q = 1$ are supposed to be given. The following standard notation will be used.

(1) The symbol $|E|$ denotes either the usual euclidean norm of E in $\mathbb{R}^N$ when $E \in \mathbb{R}^N$, or the N-dimensional Lebesgue measure of E when $E \subset \mathbb{R}^N$.
(2) For all $a > 0$, we denote $B_a := \{x \in \mathbb{R}^N; |x| < a\}$ and $B'_a := \mathbb{R}^N \setminus \overline{B}_a$.
(3) S^{N-1} will be the unit sphere in $\mathbb{R}^N$ and A^{N-1} its surface area.
(4) I_N denotes the N-dimensional identity matrix.
(5) For all open subset $\mathcal{O} \subset \mathbb{R}^N$ or $\mathcal{O} \subset \mathbb{R}$, $C_0^\infty(\mathcal{O})$ denotes the space of infinitely differentiable functions with compact support $\subset \mathcal{O}$ and $\mathcal{D}'(\mathcal{O})$ denotes the space of distributions in $\mathcal{O}$.
(6) The topological dual of a normed space $\mathcal{F}$ is denoted $\mathcal{F}^*$, and the duality pairing between $\mathcal{F}^*$ and $\mathcal{F}$ by $\langle ., . \rangle$.

Moreover, Ω is a given bounded domain of $\mathbb{R}^N$. We denote

(1) $W^{1,p}(\Omega)$ the Sobolev space defined by

$$W^{1,p}(\Omega) := \{u \in \mathcal{D}'(\Omega); u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$$

endowed with the norm

$$\|u\|_{1,p} := \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} ;$$

(2) $\mathcal{V} := W^{1,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$;

(3) $H^1(\Omega)$ the Hilbert space defined by

$$H^1(\Omega) := \left\{ u \in \mathcal{D}'(\Omega); u \in L^2(\Omega), \nabla u \in L^2(\Omega) \right\}$$

endowed with the norm

$$\|u\|_{1,2} := \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} ;$$

(4) $\mathcal{H} := H^1_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$.

2. Specific issues arising for a quasilinear elliptic equation

This section points out the specific issues arising in the process of obtaining a topological asymptotic expansion for a second order quasilinear elliptic equation, by comparison with a linear elliptic equation. We briefly recall in subsection 2.1 the main steps taken to obtain the expansion in the case of a linear elliptic equation. In subsection 2.2 we determine in the quasilinear context the conditions allowing to apply the Minty-Browder Theorem ([45], Chap. 2 §2, [27] Thm. V.15), so as to give sense to the variation of the direct state at scale 1. In the last subsection 2.4, we complete our preliminary study by announcing how three other steps of the method will have to be generalized in the quasilinear context.

In all the subsequent we shall study quasilinear elliptic equations of second order $Qu = 0$ which are Euler-Lagrange equations of functionals of the form

$$\int_\Omega \left[\gamma W(\nabla u) - f u \right], \tag{2.1}$$

where $\gamma : \Omega \to \mathbb{R}^*_+$ is a *positive conductivity function*, $W \in C^1(\mathbb{R}^N, \mathbb{R})$ is called the *potential* and $f : \Omega \to \mathbb{R}$ is a *source*. For convenience we denote the gradient field $T := \nabla W$.

As is well-known (see e.g. [32], Chap. 8), under relevant assumptions, in an appropriate functional space and with appropriate boundary condition, a function u minimizes functional (2.1) if and only if it satisfies the following Euler-Lagrange equation

$$Qu := -\operatorname{div}\left(\gamma\ T(\nabla u)\right) - f = 0. \tag{2.2}$$

For a given $p \in (2, \infty)$, we shall focus on the two following cases:

(1) the p-Laplace equation, weighted by the conductivity function γ,

$$-\operatorname{div}\left(\gamma \left|\nabla u\right|^{p-2} \nabla u\right) - f = 0, \tag{2.3}$$

which derives from the potential $W(\varphi) := \frac{1}{p} \left|\varphi\right|^p$, $\forall \varphi \in \mathbb{R}^N$;

(2) the modified p-Laplace equation

$$-\operatorname{div}\left(\gamma \left(a^2 + \left|\nabla u\right|^2\right)^{\frac{p-2}{2}} \nabla u\right) - f = 0, \tag{2.4}$$

for a given real number $a > 0$, which derives from the potential

$$W_a(\varphi) := \frac{1}{p} \left(a^2 + \left|\varphi\right|^2\right)^{\frac{p}{2}}, \quad \forall \varphi \in \mathbb{R}^N.$$

From the perspective of topological asymptotic expansions, we shall see that properties of equations (2.3) and (2.4) broadly differ, far beyond the well-known fact that equation (2.3) is degenerate while equation (2.4) is not.

2.1. **Standard steps taken for a linear elliptic equation.** The task of obtaining the topological asymptotic expansion for a linear elliptic equation was already performed many times in the literature [13, 15, 33, 35, 36, 47, 51, 59].

Let's consider here a conductivity perturbation in the Laplace equation with homogeneous Dirichlet boundary condition.

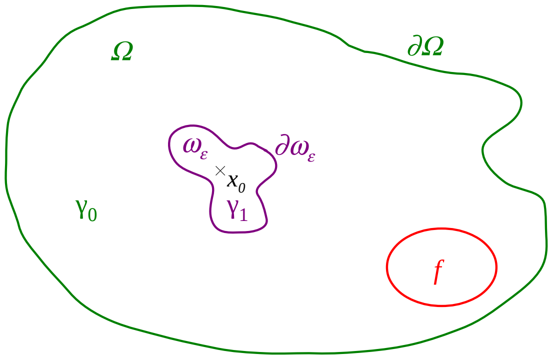

FIGURE 2. Perturbation of the conductivity in ω_ε.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$, $x_0 \in \Omega$ and a smooth bounded domain $\omega \subset \mathbb{R}^N$ such that $0 \in \omega$. For $\varepsilon > 0$, let $\omega_\varepsilon := x_0 + \varepsilon\, \omega$. For $\varepsilon > 0$ small enough it holds $\omega_\varepsilon \subset\subset \Omega$. Given two positive numbers $\gamma_0 \neq \gamma_1$ we define the *perturbed conductivity* by

$$\gamma_\varepsilon := \gamma_0 \quad \text{in } \Omega \setminus \omega_\varepsilon \text{ and } \gamma_\varepsilon := \gamma_1 \quad \text{in } \omega_\varepsilon. \tag{2.5}$$

Let a source $f \in L^2(\Omega)$ be such that $x_0 \notin \operatorname{spt}(f)$, with $\operatorname{spt}(f)$ being the support of f. The perturbed direct state u_ε is the solution of

$$\begin{cases} -\operatorname{div}(\gamma_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Let the Hilbert space $\mathcal{H} = H_0^1(\Omega)$ and $J : \mathcal{H} \to \mathbb{R}$ a Fréchet differentiable functional. The following steps are taken:

(1) By linearity, the variation $\tilde{u}_\varepsilon := u_\varepsilon - u_0$ of the direct state is simply defined by means of the Lax-Milgram theorem in the Hilbert space $\mathcal{H}$.

(2) The perturbed adjoint state v_ε is defined as the unique solution of the problem: find $v \in \mathcal{H}$ such that

$$\int_\Omega \gamma_\varepsilon \nabla v . \nabla \eta = -\langle DJ(u_0), \eta \rangle, \quad \forall \eta \in \mathcal{H},$$

where $\langle . \,, \,. \rangle$ stands for the inner product of $\mathcal{H}$.

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Thanks to the fact that the direct and adjoint states are in duality in the Hilbert space $\mathcal{H}$, one can easily transform the first order Taylor expansion of functional J and obtains

$$J(u_\varepsilon) - J(u_0) = -\int_\Omega \gamma_\varepsilon \nabla \tilde{v}_\varepsilon . \nabla \tilde{u}_\varepsilon \; + \; (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \nabla u_0 . \nabla v_0 + o\left(\|\tilde{u}_\varepsilon\|_{\mathcal{H}}\right). \tag{2.7}$$

Assuming enough regularity for $\partial\Omega$ and for the sources f and $DJ(u_0)$, one may assume (e.g. [34] Thm 8.34) that ∇u_0 and ∇v_0 are continuous in Ω, and in particular at point x_0. This entails

$$\int_{\omega_\varepsilon} \nabla v_0.\nabla u_0 = |\omega|\,\nabla v_0(x_0).\nabla u_0(x_0)\ \varepsilon^N + o(\varepsilon^N).$$

Hence, according to (2.7), the main task is to determine the asymptotic expansion of the integral:

$$\int_\Omega \gamma_\varepsilon \nabla\tilde{v}_\varepsilon.\nabla\tilde{u}_\varepsilon. \tag{2.8}$$

(3) One thus introduces the variation H of direct state (resp. the variation K of adjoint state) at scale 1 in such a way that the following approximations hold

$$\tilde{u}_\varepsilon(x) \approx \varepsilon H(\varepsilon^{-1}x) \ \text{ and } \ \tilde{v}_\varepsilon(x) \approx \varepsilon K(\varepsilon^{-1}x), \quad \text{for a.e. } x \in \Omega.$$

The conductivity γ at scale 1 is defined by

$$\gamma := \gamma_0 \ \text{ in } \mathbb{R}^N \setminus \omega \quad \text{and} \quad \gamma_\varepsilon := \gamma_1 \ \text{ in } \omega. \tag{2.9}$$

An appropriate Hilbert space $\tilde{\mathcal{H}}$ of functions defined on $\mathbb{R}^N$ is then built ([31], volume 6, chapter XI and [9]). A Poincaré inequality in $\tilde{\mathcal{H}}$ is required for coercivity. One defines H as solution of the problem:
find $H \in \tilde{\mathcal{H}}$ such that

$$\int_{\mathbb{R}^N} \gamma\nabla H.\nabla\eta + (\gamma_1-\gamma_0)\int_\omega \nabla u_0(x_0).\nabla\eta = 0, \quad \forall\eta \in \tilde{\mathcal{H}}. \tag{2.10}$$

Similarly one defines K as solution of the problem:
find $K \in \tilde{\mathcal{H}}$ such that

$$\int_{\mathbb{R}^N} \gamma\nabla K.\nabla\eta + (\gamma_1-\gamma_0)\int_\omega \nabla v_0(x_0).\nabla\eta = 0, \quad \forall\eta \in \tilde{\mathcal{H}}. \tag{2.11}$$

Again H and K are defined by means of the Lax-Milgram theorem in the Hilbert space $\tilde{\mathcal{H}}$. The integral

$$\int_{\mathbb{R}^N} \gamma\nabla K.\nabla H$$

is well defined. Plugging the test function $K \in \tilde{\mathcal{H}}$ in (2.10) and using the Green formula, one obtains

$$-\int_{\mathbb{R}^N} \gamma\nabla K.\nabla H = (\gamma_1-\gamma_0)\int_\omega \nabla u_0(x_0).\nabla K = (\gamma_1-\gamma_0)\ \nabla u_0(x_0).\int_{\partial\omega} Kn,$$

where the letter n denotes here the outward unit normal to $\partial\omega$.

Regarding the calculation of the latter integral, it follows from the linearity of equation (2.11) defining K that the mapping

$$\nabla v_0(x_0) \in \mathbb{R}^N \mapsto (\gamma_1-\gamma_0)\left[|\omega|\,\nabla v_0(x_0) + \int_{\partial\omega} Kn\right] \in \mathbb{R}^N$$

is linear. It only depends on the set ω and on the ratio γ_1/γ_0. Thus there exists an $N\times N$ matrix $\mathcal{P} = \mathcal{P}(\omega, \gamma_1/\gamma_0)$, called *polarization tensor*, such that

$$(\gamma_1-\gamma_0)\left[|\omega|\,\nabla v_0(x_0) + \int_{\partial\omega} Kn_{out}\right] \ = \ \mathcal{P}\nabla v_0(x_0).$$

Such polarization tensor can be explicitly calculated for various types of sets ω, see e.g. [6, 13, 29, 40, 55].

(4) One then needs to determine the asymptotic behaviors of variations of direct and adjoint states at scale 1. Problems (2.11) and (2.10) are two-phase transmission linear problems in $\mathbb{R}^N$ and were extensively studied, see e.g. [6] Part I. By convolution of the source with a fundamental solution of the Laplace equation, one obtains the asymptotic behavior of K and ∇K as follows:
$$K(y) = O\left(|y|^{1-N}\right) \text{ and } \nabla K(y) = O\left(|y|^{-N}\right) \text{ when } |y| \to +\infty.$$
The same asymptotic behaviors hold for H and ∇H.

(5) It matters to know whether the variation of the direct state (resp. of the adjoint state) 'far away' from the perturbation ω_ε is of a higher order, i.e. is negligible, compared to the same variation 'near' the perturbation, as shown on Figure 3. That is the reason why the asymptotic behaviors of H and K and of their gradient fields play a key role in the justification of expansion
$$\int_\Omega \gamma_\varepsilon \nabla\tilde{v}_\varepsilon . \nabla\tilde{u}_\varepsilon = \varepsilon^N \int_{\mathbb{R}^N} \gamma \nabla K . \nabla H \;+\; o(\varepsilon^N). \tag{2.12}$$

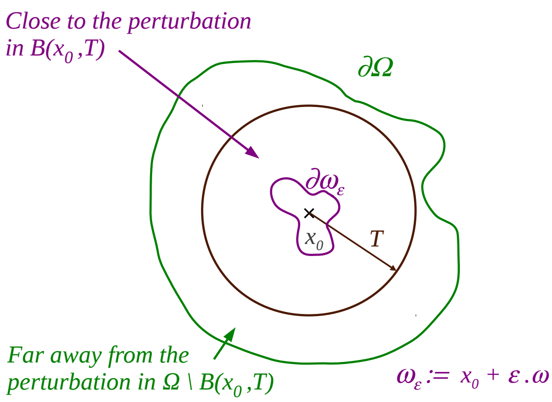

FIGURE 3. Close to the perturbation versus far away from the perturbation.

(6) From (2.7) and (2.12), it eventually follows the desired topological asymptotic expansion
$$J(u_\varepsilon) - J(u_0) = g(x_0)\, \varepsilon^N + \; o(\varepsilon^N),$$
with
$$g(x_0) = \nabla u_0(x_0). \left(\mathcal{P}\nabla v_0(x_0)\right) = \nabla u_0(x_0)^T \mathcal{P} \nabla v_0(x_0).$$

For more details we refer the reader to[13] or to [26].

2.2. **First steps taken for a quasilinear elliptic equation.** We choose to first study the case of the p-Laplace equation. For a given $f \in L^q(\Omega)$, the perturbed direct state u_ε satisfies the equation
$$\begin{cases} -\operatorname{div}\left(\gamma_\varepsilon \, |\nabla u_\varepsilon|^{p-2} \, \nabla u_\varepsilon\right) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

In the space $\mathcal{V} = W_0^{1,p}(\Omega)$, it is standard ([44] or [18] §6.6) to argue that u_ε is uniquely defined by
$$\{u_\varepsilon\} = \operatorname*{argmin}_{u \in \mathcal{V}} \left\{ \int_\Omega \frac{\gamma_\varepsilon}{p} |\nabla u|^p - f u \right\},$$

and that equivalently u_ε is the unique solution of the problem:
find $u \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon \, |\nabla u|^{p-2} \, \nabla u.\nabla \eta = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V}. \tag{2.13}$$

Denote u_0 the unperturbed direct state and $\tilde{u}_\varepsilon := u_\varepsilon - u_0$ the variation of the direct state. Hence calculating the difference between equation (2.13) and the equation satisfied by u_0, one obtains that function $\tilde{u}_\varepsilon$ solves the problem:

find $\tilde{u} \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon \left[|\nabla u_0 + \nabla \tilde{u}|^{p-2} (\nabla u_0 + \nabla \tilde{u}) - |\nabla u_0|^{p-2} \, \nabla u_0 \right].\nabla \eta \\ + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} |\nabla u_0|^{p-2} \, \nabla u_0.\nabla \eta = 0, \quad \forall \eta \in \mathcal{V}. \tag{2.14}$$

Conversely consider equation (2.14) and the nonlinear operator $A_\varepsilon : \mathcal{V} \to \mathcal{V}^*$ defined by

$$\langle A_\varepsilon \tilde{u}, \eta \rangle := \int_\Omega \gamma_\varepsilon \left[|\nabla u_0 + \nabla \tilde{u}|^{p-2} (\nabla u_0 + \nabla \tilde{u}) - |\nabla u_0|^{p-2} \, \nabla u_0 \right].\nabla \eta, \quad \forall \tilde{u}, \eta \in \mathcal{V}.$$

Due to Hölder's inequality, it is clear that A_ε is well defined. Moreover, as shown below, the Minty-Browder theorem ([27], Theorem 5.15) can be applied to operator A_ε so as to prove that equation (2.14) admits a unique solution in $\mathcal{V}$. Due to uniqueness, the latter solution equals $\tilde{u}_\varepsilon$.

Let's sketch briefly the arguments showing that A_ε satisfies the assumptions required by the Minty-Browder theorem (similar arguments will be detailed later on in the proof of Proposition 3.7).

(1) The continuity of A_ε is based on the following inequality ([44], p.73):

$$\left| |\varphi + \psi|^{p-2} (\varphi + \psi) - |\varphi|^{p-2} \varphi \right| \le (p-1) \, |\psi| \int_0^1 |\varphi + t\psi|^{p-2} \; dt \\ \le 2^{p-2}(p-1) \, |\psi| \left(|\varphi|^{p-2} + |\psi|^{p-2} \right) \quad \forall \varphi, \psi \in \mathbb{R}^N. \tag{2.15}$$

It follows from (2.15) and from Hölder's inequality that, for all $u, v, \eta \in \mathcal{V}$,

$$|\langle A_\varepsilon(u+v) - A_\varepsilon(u), \eta \rangle| \le C \int_\Omega \left[|\nabla(u_0 + u)|^{p-2} \, |\nabla v| + |\nabla v|^{p-1} \right] |\nabla \eta| \\ \le C \left[\|\nabla(u_0 + u)\|^{p-2}_{L^p(\Omega)} \, \|\nabla v\|_{L^p(\Omega)} + \|\nabla v\|^{\frac{p}{q}}_{L^p(\Omega)} \right] \|\nabla \eta\|_{L^p(\Omega)} \,,$$

with $C = 2^{p-2}(p-1)\max(\gamma_0, \gamma_1)$. Hence for all $u, v \in \mathcal{V}$, it holds

$$\|A_\varepsilon(u+v) - A_\varepsilon(u)\|_{\mathcal{V}^*} \le C \left[\|\nabla(u_0 + u)\|^{p-2}_{L^p(\Omega)} \, \|\nabla v\|_{L^p(\Omega)} + \|\nabla v\|^{\frac{p}{q}}_{L^p(\Omega)} \right].$$

The continuity of A_ε follows.

(2) As Ω is bounded, by the Poincaré inequality, the norm $\|.\|_{\mathcal{V}}$ is equivalent to the semi-norm $|.|_{\mathcal{V}}$ in $\mathcal{V}$. Moreover the following p-ellipticity inequality holds ([44], page 71 (I)):
for all $p \in (2, \infty)$, there exists $c = c(p) > 0$ such that

$$\left[|\varphi + \psi|^{p-2} (\varphi + \psi) - |\varphi|^{p-2} \varphi \right].\psi \ge c \; |\psi|^p \,, \quad \forall \; \varphi, \psi \in \mathbb{R}^N. \tag{2.16}$$

The strict monotonicity and the coercivity of A_ε follow.

We now prepare the change of scale by taking an intermediate step. Assume that u_0 is regular enough and denote $U_0 := \nabla u_0(x_0)$ its gradient at the center x_0 of perturbation. We first approximate the variation $\tilde{u}_\varepsilon$ by the function h_ε solution of:

find $h \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon \left[|U_0 + \nabla h|^{p-2} (U_0 + \nabla h) - |U_0|^{p-2} U_0 \right] .\nabla\eta \\ + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} |U_0|^{p-2} U_0 .\nabla\eta = 0, \quad \forall \eta \in \mathcal{V}.$$

Similarly the Minty-Browder theorem ensures the existence and uniqueness of the solution.

Moving now to scale 1, we look for a function space $\mathcal{W}(\mathbb{R}^N)$ in which one can apply the Minty-Browder theorem to the problem:

find $H \in \mathcal{W}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla H|^{p-2} (U_0 + \nabla H) - |U_0|^{p-2} U_0 \right] .\nabla\eta \\ + (\gamma_1 - \gamma_0) \int_{\omega} |U_0|^{p-2} U_0 .\nabla\eta = 0 \quad \forall \eta \in \mathcal{W}(\mathbb{R}^N), \quad (2.17)$$

where the perturbed conductivity γ at scale 1 is still defined by (2.9).

For that purpose, we exclude the trivial case $U_0 = 0$. We need to find a reflexive Banach space $\mathcal{W}(\mathbb{R}^N)$ such that the nonlinear operator $A : \mathcal{W}(\mathbb{R}^N) \to \mathcal{W}^*(\mathbb{R}^N)$ defined by

$$\langle Au, \eta \rangle := \int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla u|^{p-2} (U_0 + \nabla u) - |U_0|^{p-2} U_0 \right] .\nabla\eta, \quad \forall u, \eta \in \mathcal{W}(\mathbb{R}^N)$$

is well defined in $\mathcal{W}(\mathbb{R}^N)$ and satisfies all the assumptions required by the Minty-Browder theorem.

2.3. **A two-norm discrepancy between $L^p(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$.** From (2.16) we infer

$$\langle Au, u \rangle \;=\; \int_{\mathbb{R}^N} \gamma \left[|U_0 + \nabla u|^{p-2} (U_0 + \nabla u) - |U_0|^{p-2} U_0 \right] .\nabla u \;\geq\; c' \left\| \nabla u \right\|^p_{L^p(\mathbb{R}^N)}, \quad (2.18)$$

with $c' = c\ \min(\gamma_0, \gamma_1) > 0$. Therefore the coercivity of A could be secured in $\mathcal{W}(\mathbb{R}^N)$ provided that $u \in \mathcal{W}(\mathbb{R}^N) \Rightarrow \nabla u \in L^p(\mathbb{R}^N)$ and should an equivalence hold in $\mathcal{W}(\mathbb{R}^N)$ between the norm $\|u\|_{\mathcal{W}(\mathbb{R}^N)}$ and the semi-norm $|u|_{\mathcal{W}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)}$.

To obtain such an equivalence of the norm with the semi-norm in an unbounded domain (e.g. [31], volume 6, chapter XI and [9], Annexe A), a classical approach would be to define $\mathcal{W}(\mathbb{R}^N)$ as the quotient space

$$\mathcal{W}(\mathbb{R}^N) = \mathcal{W}^w(\mathbb{R}^N)/P_p,$$

where $\mathcal{W}^w(\mathbb{R}^N)$ is a weighted Sobolev space of the type

$$\mathcal{W}^w(\mathbb{R}^N) := \left\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w_p u \in L^p(\mathbb{R}^N) \text{ and } \nabla u \in L^p(\mathbb{R}^N) \right\},$$

for an appropriate weight $w_p : \mathbb{R}^N \to \mathbb{R}_+$, and P_p is a space of polynomials belonging to $\mathcal{W}^w(\mathbb{R}^N)$. In fact here, P_p can only be $\{0\}$ or $\mathbb{R}$, the set of constants. The weighted space $\mathcal{W}^w(\mathbb{R}^N)$ would then be equipped with the norm

$$\|u\|_{\mathcal{W}^w(\mathbb{R}^N)} := \|w_p u\|_{L^p(\mathbb{R}^N)} \;+\; \|\nabla u\|_{L^p(\mathbb{R}^N)},$$

and its quotient space $\mathcal{W}(\mathbb{R}^N)$ with the norm ([28], 11.2)

$$\|u\|_{\mathcal{W}(\mathbb{R}^N)} := \inf_{m \in P_p} \|w_p (u + m)\|_{L^p(\mathbb{R}^N)} \;+\; \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Suppose that we have built such a quotient space. It follows from inequality (2.15) that for all $u, \eta \in \mathcal{W}(\mathbb{R}^N)$

$$|\langle Au, \eta\rangle| \leq C \int_{\mathbb{R}^N} |\nabla u| \left(|U_0|^{p-2} + |\nabla u|^{p-2}\right) |\nabla \eta|$$

with $C = 2^{p-2}(p-1)\max(\gamma_0, \gamma_1)$. This does not guarantee that A is well defined in all the space $\mathcal{W}(\mathbb{R}^N)$. Consider the subspace

$$\mathcal{V}(\mathbb{R}^N) := \left\{u \in \mathcal{W}(\mathbb{R}^N); \nabla u \in L^2(\mathbb{R}^N)\right\}$$

endowed with the norm

$$\|u\|_{\mathcal{V}(\mathbb{R}^N)} := \inf_{m \in P_p} \|w_p(u+m)\|_{L^p(\mathbb{R}^N)} \; + \; \|\nabla u\|_{L^p(\mathbb{R}^N)} + \; \|\nabla u\|_{L^2(\mathbb{R}^N)} \,. \tag{2.19}$$

For all $u, \eta \in \mathcal{V}(\mathbb{R}^N)$ the Cauchy-Schwarz and Hölder inequalities entail

$$|\langle Au, \eta\rangle| \leq C \left(|U_0|^{p-2} \|\nabla u\|_{L^2(\mathbb{R}^N)} \|\nabla \eta\|_{L^2(\mathbb{R}^N)} + \|\nabla u\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)} \|\nabla \eta\|_{L^p(\mathbb{R}^N)} \right). \tag{2.20}$$

Thus A is well defined in the subspace $\mathcal{V}(\mathbb{R}^N)$. In addition, A is a bounded linear operator $\mathcal{V}(\mathbb{R}^N) \to \mathcal{V}^*(\mathbb{R}^N)$. Nevertheless inequality (2.18) shows that one cannot expect A to be coercive in $\mathcal{V}(\mathbb{R}^N)$ equipped with the norm (2.19), as $\langle Au, u\rangle$ does not provide control over the term $\|\nabla u\|_{L^2(\mathbb{R}^N)}$.

In comparison with the method recalled in section 2.1, it thus appears that the step of defining the variation of the direct state at scale 1 by means of the Minty-Browder theorem, requires both

- to consider a functional space whose norm gives control on both the L^p and the L^2 norms of the gradient, and which in addition enjoys a Poincaré inequality;
- to consider a quasilinear elliptic operator A enjoying both p- and 2- ellipticity properties, which is not the case for the p-Laplacian.

The first requirement justifies our construction of the Banach space $\mathcal{V}(\mathbb{R}^N)$ (and the Hilbert space $\mathcal{H}(\mathbb{R}^N)$ when $p = 2$) in appendix A. The second requirement provides restrictions on the class of quasilinear equations we address in section 3.

This situation is similar to the two-norm discrepancy known since the 1970's in the stability analysis of nonlinear optimal control [5, 39, 43, 48], where the Lagrangian is typically twice differentiable in L^∞ norm but only coercive in L^2 norm.

2.4. **Other changes in the quasilinear context.** Several other steps have to be generalized in order to obtain the topological asymptotic expansion in the quasilinear context. In particular we shall have to:

(1) ensure duality between the variation of the direct state and the corresponding variation of the adjoint state at each step of approximation. This task, straightforward in the linear case within the framework of Hilbert spaces, will be made again possible by considering both relevant embeddings and *a posteriori* regularity properties.
(2) determine the asymptotic behavior in $\mathbb{R}^N$ of the variation of the direct state at scale 1. This function will be solution of a nonlinear transmission problem in $\mathcal{V}(\mathbb{R}^n)$, for which techniques of convolution do not apply. We shall build a supersolution and a subsolution and then prove a comparison theorem.
(3) determine with respect to the variation of the direct state, what does mean 'far away from the perturbation' by opposition to 'close to the perturbation'. This question will be dealt with in Propositions 3.15 and 3.16.

3. Topological asymptotic expansion for a class of quasilinear elliptic equations

3.1. **A class of non-quadratic potentials.** Let $W : \mathbb{R}^N \to \mathbb{R}$ be a twice Fréchet-differentiable function. Denote $T : \mathbb{R}^N \to \mathbb{R}^N$ the gradient field $T := \nabla W$. At the next order of derivation, for all $\varphi \in \mathbb{R}^N$, we define $S_\varphi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$S_\varphi(\psi) := T(\varphi + \psi) - T(\varphi) - DT(\varphi).\psi, \quad \forall\, \psi \in \mathbb{R}^N.$$

In view of the arguments expounded in section 2, we make:

Assumption 3.1. The potential W satisfies the following conditions.

(1) For some $\alpha > 0$, it holds $W \in C^{2,\alpha}(\mathbb{R}^N, \mathbb{R})$.

(2) There exist $b_0 > a_0 > 0$ such that

$$a_0\, |\varphi|^p \leq W(\varphi) \leq b_0(1 + |\varphi|^p), \quad \forall\, \varphi \in \mathbb{R}^N.$$

(3) There exists $a_1 > 0$ such that

$$|T(\varphi)| \leq a_1\, |\varphi|\, (1 + |\varphi|^{p-2}), \quad \forall\, \varphi \in \mathbb{R}^N.$$

(4) There exist $0 < c < C$ such that

$$c(1 + |\varphi|^2)^{\frac{p-2}{2}}\, |\psi|^2 \leq DT(\varphi)\psi.\psi \leq C(1 + |\varphi|^2)^{\frac{p-2}{2}}\, |\psi|^2\,, \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$

(5) There exists $c > 0$ such that

$$\left(T(\varphi + \psi) - T(\varphi)\right).\psi \geq c(|\psi|^p + |\psi|^2), \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$

(6) There exists $C > 0$ such that

$$|T(\varphi + \psi) - T(\varphi)| \leq C\, |\psi| \left[1 + |\varphi|^{p-2} + |\psi|^{p-2}\right], \quad \forall\, \varphi, \psi \in \mathbb{R}^N.$$

(7) For any $M > 0$ there exist $c_0 = c_0(M, p) \geq 0$ and $c_{p-3} = c_{p-3}(p) \geq 0$ such that

$$|S_\varphi(\psi_2) - S_\varphi(\psi_1)| \leq |\psi_2 - \psi_1|\, (|\psi_1| + |\psi_2|) \left[c_0 + c_{p-3}\, (|\psi_1| + |\psi_2|)^{p-3}\right],$$
$$\forall \varphi \in B(0, M),\ \forall \psi_1, \psi_2 \in \mathbb{R}^N.$$

In addition the constant c_{p-3} satisfies $c_{p-3} = 0,\ \forall p \in [2, 3]$.

(8) For any $M > 0$ there exist $d_0 = d_0(M, p) \geq 0$ and $d_{p-4} = d_{p-4}(p) \geq 0$ such that

$$|S_{\varphi_2}(\psi) - S_{\varphi_1}(\psi)| \leq |\varphi_2 - \varphi_1|\, |\psi|^2 \left[d_0 + d_{p-4}\, |\psi|^{p-4}\right], \quad \forall \varphi_1, \varphi_2 \in B(0, M),\ \forall \psi \in \mathbb{R}^N.$$

In addition the constant d_{p-4} satisfies $d_{p-4} = 0,\ \forall p \in [2, 4]$.

Let us comment on the conditions stated in Assumption 3.1.

(1) Conditions (2), (3) and (4) are classical growth conditions about respectively the potential W, the gradient field T and the Hessian DT, in works related to quasilinear elliptic equations (e.g. [34, 46]). Condition (4) entails that potential W is strictly convex and provides 2-ellipticity to variational problems defining the adjoint state and its variations.

(2) Condition (5) ensures the combined p- and 2-ellipticity property.

(3) Condition (6) will be essential to estimate the variations of the direct state at various steps of approximation. When φ is bounded it implies:
for any $M > 0$ there exist $b_1 > 0$ and $b_{p-1} > 0$ such that

$$|T(\varphi + \psi) - T(\varphi)| \leq b_1\, |\psi| + b_{p-1}\, |\psi|^{p-1}\,, \quad \forall\, \varphi \in B(0, M), \forall\, \psi \in \mathbb{R}^N. \tag{3.1}$$

Note that, should we have made the additional assumption that $T(0) = 0$, then condition (6) would have implied condition (3).

(4) Condition (7) gives control over nonlinearity of gradient field T at a given point φ and is related to the third derivative of W, if it exists. When φ is bounded it implies: for any $M > 0$ there exist two constants $c_0 \geq 0$ and $c_{p-3} \geq 0$ satisfying nullity condition stated in condition (7), such that

$$|S_\varphi(\psi)| \leq c_0 \, |\psi|^2 + c_{p-3} \, |\psi|^{p-1} \quad \forall \varphi \in B(0, M), \forall \psi \in \mathbb{R}^N. \tag{3.2}$$

(5) Condition (8) takes into account the fourth derivative of W, if it exists, and accounts for the variation of the nonlinearity of gradient field T from a given point φ_1 to another point φ_2.

In the subsequent we shall only write 'condition (i)' instead of 'condition (i) of Assumption 3.1' whenever no confusion is possible.

The class of potentials satisfying Assumption 3.1 encompasses the archetype of non-degenerate elliptic potentials ([34] p.261 or [63] p.343) given, for some $a > 0$, by

$$W_a \quad : \quad \varphi \in \mathbb{R}^N \mapsto \frac{1}{p} \left(a^2 + |\varphi|^2 \right)^{p/2}. \tag{3.3}$$

For potential W_a, the p- and 2-ellipticity property required by condition (5) follows from a slightly extended version of an inequality given by Lindqvist in [44], page 71 (I).

Proposition 3.2. *Let $a > 0$ and $p \in [2, \infty)$. Then there exists $c > 0$ such that*

$$\left[(a^2 + |\varphi + \psi|^2)^{\frac{p-2}{2}} (\varphi + \psi) - (a^2 + |\varphi|^2)^{\frac{p-2}{2}} \varphi \right] . \psi \geq c \; (|\psi|^p + |\psi|^2), \quad \forall \; \varphi, \psi \in \mathbb{R}^N.$$

The proof is deferred to subsection 4.1 on page 23. At the price of some calculations we obtain the following result, whose proof is given in subsection 4.2 on page 24.

Proposition 3.3. *Let $a > 0$ and $p \in [2, \infty)$. Then potential W_a satisfies Assumption 3.1.*

3.2. **The perturbed quasilinear equation.** We assume that Ω is a bounded domain of $\mathbb{R}^N$ with C^2-boundary $\partial\Omega$. Let a function $f \in C^{0,\alpha}(\Omega)$, for some $\alpha > 0$, with support $\mathrm{spt}(f) \subset\subset \Omega$. A point $x_0 \in \Omega \setminus \mathrm{spt}(f)$ is given. Without loss of generality we assume that $x_0 = 0$.

Consider a bounded domain ω of $\mathbb{R}^N$ with a C^2-boundary $\partial\omega$ such that $0 \in \omega$. For all $\varepsilon \geq 0$, let $\omega_\varepsilon := \varepsilon\omega$. In all this chapter, we assume $\varepsilon \geq 0$ is small enough such that $\omega_\varepsilon \subset\subset \Omega \setminus spt(f)$. Moreover changing if appropriate ω (resp. ε) into ω/λ (resp. into $\lambda\varepsilon$) for some $\lambda > 0$ large enough, we can assume without loss of generality that there exist two real numbers

$$0 < \rho < R \quad \text{such that} \quad \omega \subset\subset B(0, \rho) \subset B(0, R) \subset\subset \Omega \setminus \mathrm{spt}(f). \tag{3.4}$$

Define the perturbed conductivity function $\gamma_\varepsilon : \Omega \to \mathbb{R}$ by

$$\gamma_\varepsilon := \gamma_0 \text{ in } \Omega \setminus \omega_\varepsilon \;\; \text{and} \;\; \gamma_\varepsilon := \gamma_1 \text{ in } \omega_\varepsilon, \tag{3.5}$$

where γ_0, γ_1 are two positive real numbers with $\gamma_0 \neq \gamma_1$. Denote $\underline{\gamma} := \min(\gamma_0, \gamma_1) (> 0)$ and $\overline{\gamma} := \max(\gamma_0, \gamma_1)$.

We define the direct state u_ε in the Banach space $\mathcal{V} := W_0^{1,p}(\Omega)$.

Lemma 3.4. *For all $\varepsilon \geq 0$ small enough, the functional*

$$\mathcal{W}_\varepsilon : \eta \in \mathcal{V} \mapsto \int_\Omega \gamma_\varepsilon W(\nabla\eta) \; - \; \int_\Omega f\eta$$

is continuous, strictly convex and coercive in $\mathcal{V}$. We define u_ε as

$$\{u_\varepsilon\} = \underset{\eta \in \mathcal{V}}{\operatorname{argmin}} \, \mathcal{W}_\varepsilon(\eta).$$

This solution is characterized by the Euler-Lagrange equation:

$$\textit{find } u_\varepsilon \in \mathcal{V} \textit{ such that } \int_\Omega \gamma_\varepsilon T(\nabla u_\varepsilon).\nabla\eta = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V}, \tag{3.6}$$

whose strong form is

$$\textit{find } u_\varepsilon \in W^{1,p}(\Omega) \textit{ such that} \begin{cases} -\operatorname{div}\left(\gamma_\varepsilon T(\nabla u_\varepsilon)\right) = f & \textit{in } \Omega, \\ u_\varepsilon = 0 & \textit{on } \partial\Omega. \end{cases}$$

The proof, based on standard arguments, is sketched in subsection 4.3 on page 27.

3.3. **Topological asymptotic expansion.** For all $\varepsilon \geq 0$ small enough, consider a functional $\mathcal{J}_\varepsilon : \mathcal{H} \to \mathbb{R}$ such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) + \langle G, u_\varepsilon - u_0 \rangle + \delta_2\, \varepsilon^N + R(\varepsilon), \tag{3.7}$$

where

(1) G denotes a bounded linear form on $\mathcal{H}$;
(2) $\delta_2 \in \mathbb{R}$;
(3) the remainder $R(\varepsilon)$ is
   (a) either of the form

$$R(\varepsilon) = o\left(\|u_\varepsilon - u_0\|_{\mathcal{H}}^2 \right), \tag{3.8}$$

   (b) or of the form

$$R(\varepsilon) = O\left(\int_{\Omega \setminus B(0, \tilde{\alpha}\varepsilon^{\tilde{r}})} |\nabla(u_\varepsilon - u_0)|^p + |\nabla(u_\varepsilon - u_0)|^2 \right), \tag{3.9}$$

   for a given $\tilde{\alpha} > 0$ and a given $\tilde{r} \in (0,1)$. In this case, the remainder $R(\varepsilon)$ is controlled by the variation of the direct state 'far away' from the perturbation.

A classical example of such a functional is given by the compliance

$$u \in \mathcal{H} \mapsto J(u) = \int_\Omega f u, \tag{3.10}$$

with in this particular case, $G = f$ and $\delta_2 = 0$.

We now have all the ingredients to state our main result which provides the topological asymptotic expansion of $\mathcal{J}_\varepsilon(u_\varepsilon)$. We denote:

- by u_0 the unperturbed direct state defined by (3.6) in the case $\varepsilon = 0$;
- by $U_0 := \nabla u_0(x_0)$ the gradient of u_0 at point x_0;
- by H the variation of the direct state at scale 1 in $\mathbb{R}^N$ defined by (3.19);
- by v_0 the unperturbed adjoint state defined by (3.47) in the case $\varepsilon = 0$;
- by $V_0 := \nabla v_0(x_0)$ the gradient of v_0 at point x_0;
- by K the variation of the adjoint state at scale 1 in $\mathbb{R}^N$ defined by (3.50);
- by γ the conductivity function at scale 1 defined by

$$\gamma := \gamma_0 \text{ in } \mathbb{R}^N \setminus \omega \quad \text{ and } \quad \gamma := \gamma_1 \text{ in } \omega; \tag{3.11}$$

- by $\mathcal{P}$ the polarization tensor defined by (3.70), and which only depends on the set ω, on the definite positive matrix $DT(U_0)$ and on the ratio γ_1/γ_0.

Theorem 3.5. *Assume that*

- *the potential W satisfies Assumption 3.1;*
- *the functional $\mathcal{J}_\varepsilon$ satisfies an expansion of the type (3.7);*
- *the direct unperturbed state satisfies $u_0 \in L^\infty(\Omega)$;*
- *the unperturbed adjoint state satisfies $v_0 \in L^\infty(\Omega)$, $\nabla v_0 \in L^\infty(\Omega)$ and ∇v_0 is Hölder continuous in a neighborhood of x_0;*
- *the variation of the direct state at scale 1 satisfies the asymptotic behavior stated in Assumption 3.13.*

Then for $\varepsilon > 0$ small enough it holds

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = \varepsilon^N g(x_0) + o(\varepsilon^N), \tag{3.12}$$

where the topoogical gradient is given by

$$g(x_0) \;:=\; T(U_0)^T \mathcal{P} V_0 + \delta_2 \tag{3.13}$$

$$+ \int_{\mathbb{R}^N} \gamma S_{U_0}(\nabla H).\,(V_0 + \nabla K)\,. \tag{3.14}$$

Two terms (3.13) and (3.14) appear in the formula on the topological gradient.

- In the linear case, where $S_{U_0} = 0$, the topological gradient $g(x_0)$ reduces to the first term (3.13). At least when $\delta_2 = 0$, it can be computed at every point $x_0 \in \Omega$ with the only knowledge of the fields ∇u_0, ∇v_0 and the polarization tensor $\mathcal{P}$, which however also depends on $DT(\nabla u_0)$. A number of closed formulas for the polarization tensor can be found e.g. in [7, 13, 33, 40].
- The term (3.14) appears here for the first time. It accounts for the component of the topological gradient which is caused by the nonlinearity of the equation.

It is worth emphasizing that some of the regularity assumptions made in Theorem 3.5 can be directly ascertained in some cases as follows.

- In the prototype case $\omega = B(0,1)$, $W = W_a$ for some $a > 0$ and $\gamma_1 < \gamma_0$, define
$$\bar{p} := 2 + \left(1 + \frac{a^2}{|U_0|^2}\right)\frac{N}{N-2},$$
with the convention that $\bar{p} = +\infty$ when $N = 2$. If $p \in [2, \bar{p})$, then no assumption has to be made about the asymptotic behavior of function H, as it is then ensured by virtue of the Theorem 3.12 stated hereafter on page 18.
- The assumption $u_0 \in L^\infty(\Omega)$ is theoretically needed for proving the $C^{1,\beta}(\overline{\Omega})$ regularity of u_0 (see Lemma 3.6). In practice this assumption can be taken for granted.
- When G is regular enough, Lemma 3.17 states that v_0 in $C^{1,\tilde{\beta}}(\overline{\Omega})$. Hence no assumption is needed about the regularity of v_0.

So as to prove Theorem (3.5), we shall now study the variation of the direct state in section 3.4, the variation of the adjoint state in section 3.5 and lastly the asymptotic behavior of $\mathcal{J}_\varepsilon(u_\varepsilon)$ in section 3.6.

3.4. **Variation of the direct state.**

3.4.1. *About the regularity of the unperturbed direct state.* In the unperturbed case $\varepsilon = 0$, Euler-Lagrange equation (3.6) reads

$$\int_\Omega \gamma_0 T(\nabla u_0).\nabla\eta = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V}. \tag{3.15}$$

Lemma 3.6. *Assume that $u_0 \in L^\infty(\Omega)$. Then it holds $u_0 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$.*

The proof is available in subsection 4.4 on page 28.

3.4.2. *Step 1: variation $u_\varepsilon - u_0$.* Let $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in \mathcal{V}$. After (3.15), it is straightforward from Lemma 3.4 that $\tilde{u}_\varepsilon$ is characterized by the Euler-Lagrange equation: find $\tilde{u} \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon T(\nabla u_0 + \nabla\tilde{u}).\nabla\eta = \int_\Omega \gamma_0 T(\nabla u_0).\nabla\eta, \quad \forall \eta \in \mathcal{V}, \tag{3.16}$$

Since $\gamma_\varepsilon - \gamma_0 = \gamma_1 - \gamma_0$ in ω_ε and $\gamma_\varepsilon - \gamma_0 = 0$ in $\Omega \setminus \omega_\varepsilon$, the latter can be rewritten

$$\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla\tilde{u}_\varepsilon) - T(\nabla u_0)\right].\nabla\eta \;+\; (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\eta = 0, \quad \forall \eta \in \mathcal{V}. \tag{3.17}$$

3.4.3. *Step* 2*: approximation of variation* $\tilde{u}_\varepsilon$*.* We approximate $\tilde{u}_\varepsilon$ by the solution h_ε of the following Euler-Lagrange equation: find $h \in \mathcal{V}$ such that

$$\int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla h) - T(U_0)\right].\nabla\eta \ + \ (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(U_0).\nabla\eta = 0, \quad \forall \eta \in \mathcal{V}. \tag{3.18}$$

3.4.4. *Step* 3*: change to scale* 1*.* We look for a function H which may approximate the map $y \in \Omega/\varepsilon \mapsto \varepsilon^{-1} h_\varepsilon(\varepsilon y)$.

With our set of assumptions, there is no a priori guarantee that $W(\nabla H)$ is integrable over $\mathbb{R}^N$. For instance, in the case of W_a, it holds $W_a(\psi) \geq \frac{1}{p}a^p > 0$. Hence there is no hope to define function H as the minimizer of a functional involving the integral $\int_{\mathbb{R}^N} \gamma W(\nabla H)$. Instead we start from the variational form (3.18) by applying the Minty-Browder theorem in the reflexive Banach space $\mathcal{V}(\mathbb{R}^N)$ (see appendix A).

Proposition 3.7. *There exists a unique function $H \in \mathcal{V}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla H) - T(U_0)\right].\nabla\eta = -(\gamma_1 - \gamma_0)\int_\omega T(U_0).\nabla\eta, \quad \forall \eta \in \mathcal{V}(\mathbb{R}^N). \tag{3.19}$$

Proof. Using inequality (3.1) and Hölder's inequality, we obtain that for all $\eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\gamma \left[T(U_0 + \nabla\eta_1) - T(U_0)\right].\nabla\eta_2\right| &\leq \int_{\mathbb{R}^N} \overline{\gamma} \ \left(b_1 \left|\nabla\eta_1\right| + b_{p-1}\left|\nabla\eta_1\right|^{p-1}\right)\left|\nabla\eta_2\right| \\ &\leq \overline{\gamma} b_1 \left\|\nabla\eta_1\right\|_{L^2(\mathbb{R}^N)} \left\|\nabla\eta_2\right\|_{L^2(\mathbb{R}^N)} + \overline{\gamma} b_{p-1}\left\|\nabla\eta_1\right\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)} \left\|\nabla\eta_2\right\|_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{3.20}$$

Define operator A by

$$\langle A\eta_1, \eta_2\rangle := \int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla\eta_1) - T(U_0)\right].\nabla\eta_2, \quad \forall \ \eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N). \tag{3.21}$$

According to inequality (3.20), $\langle A\eta_1, \eta_2\rangle$ is well defined for all $\eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$. Moreover for all $\eta_1 \in \mathcal{V}(\mathbb{R}^N)$, it holds $A\eta_1 \in \mathcal{V}^*(\mathbb{R}^N)$ with

$$\left\|A\eta_1\right\|_{\mathcal{V}^*(\mathbb{R}^N)} \leq \overline{\gamma}\left(b_1 \left\|\nabla\eta_1\right\|_{L^2(\mathbb{R}^N)} + b_{p-1}\left\|\nabla\eta_1\right\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)}\right).$$

Then define $L \in \mathcal{V}^*(\mathbb{R}^N)$ by

$$L : \eta \in \mathcal{V}(\mathbb{R}^N) \mapsto -(\gamma_1 - \gamma_0)\int_\omega U_0.\nabla\eta.$$

The variational problem (3.19) can be equivalently written: find $H \in \mathcal{V}(\mathbb{R}^N)$ such that $AH = L$. Let us check the assumptions required by the Minty-Browder theorem.

(1) Let $\eta_1 \in \mathcal{V}(\mathbb{R}^N)$. According to condition (6) and by Hölder's inequality, it holds for all $\eta, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$

$$\begin{aligned}\left|\langle\left[A(\eta_1 + \eta) - A\eta_1\right], \eta_2\rangle\right| &= \left|\int_{\mathbb{R}^N} \gamma \left[T(U_0 + \nabla(\eta_1 + \eta)) - T(U_0 + \nabla\eta_1)\right].\nabla\eta_2\right| \\ &\leq \overline{C}\int_{\mathbb{R}^N} \left|\nabla\eta\right| \left[1 + \left|U_0 + \nabla\eta_1\right|^{p-2} + \left|\nabla\eta\right|^{p-2}\right]\left|\nabla\eta_2\right| \\ &\leq \overline{C}\int_{\mathbb{R}^N} \left|\nabla\eta\right| \left[\left(1 + 2^{p-2}\left|U_0\right|^{p-2}\right) + 2^{p-2}\left|\nabla\eta_1\right|^{p-2} + \left|\nabla\eta\right|^{p-2}\right]\left|\nabla\eta_2\right| \\ &\leq \overline{C}\left(1 + 2^{p-2}\left|U_0\right|^{p-2}\right)\left\|\nabla\eta\right\|_{L^2(\mathbb{R}^N)}\left\|\nabla\eta_2\right\|_{L^2(\mathbb{R}^N)} \\ &\quad + \ \overline{C}\left[2^{p-2}\left\|\nabla\eta_1\right\|^{p-2}_{L^p(\mathbb{R}^N)}\left\|\nabla\eta\right\|_{L^p(\mathbb{R}^N)} + \left\|\nabla\eta\right\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)}\right]\left\|\nabla\eta_2\right\|_{L^p(\mathbb{R}^N)},\end{aligned}$$

where $\overline{C} := C\,\overline{\gamma}$ and C is the constant of condition (6). Hence

$$\|A(\eta_1+\eta)-A\eta_1\|_{\mathcal{V}^*(\mathbb{R}^N)} \leq \tilde{C}\left[\|\nabla\eta\|_{L^2(\mathbb{R}^N)} + \|\nabla\eta_1\|^{p-2}_{L^p(\mathbb{R}^N)}\|\nabla\eta\|_{L^p(\mathbb{R}^N)} + \|\nabla\eta\|^{\frac{p}{q}}_{L^p(\mathbb{R}^N)}\right],$$

where $\tilde{C} := \overline{C}\max\left(1+2^{p-2}\left|U_0\right|^{p-2}, 2^{p-2}\right)$. It follows that $A(\eta_1+\eta)-A\eta_1 \to 0$ in $\mathcal{V}^*(\mathbb{R}^N)$ when $\eta \to 0$ in $\mathcal{V}(\mathbb{R}^N)$. Thus A is continuous at point η_1, for all $\eta_1 \in \mathcal{V}(\mathbb{R}^N)$.

(2) According to condition (5), there exists $c > 0$ such that for all $\eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N)$,

$$\begin{aligned}\langle A\eta_1 - A\eta_2, \eta_1-\eta_2\rangle &= \int_{\mathbb{R}^N}\gamma\left[T(U_0+\nabla\eta_1)-T(U_0+\nabla\eta_2)\right].(\nabla\eta_1-\nabla\eta_2)\\ &\geq c\underline{\gamma}\left(\|\nabla\eta_1-\nabla\eta_2\|^p_{L^p(\mathbb{R}^N)} + \|\nabla\eta_1-\nabla\eta_2\|^2_{L^2(\mathbb{R}^N)}\right).\end{aligned}$$

Hence

$$\langle A\eta_1 - A\eta_2, \eta_1-\eta_2\rangle \geq 0, \quad \forall \eta_1,\eta_2 \in \mathcal{V}(\mathbb{R}^N).$$

In addition, if

$$\langle A\eta_1 - A\eta_2, \eta_1-\eta_2\rangle = 0$$

then $\nabla\eta_1 = \nabla\eta_2$ a.e. in $\mathbb{R}^N$, and thus $\eta_1 = \eta_2$ in the quotient space $\mathcal{V}(\mathbb{R}^N)$. Hence A is strictly monotone.

(3) Lastly, according to condition (5), there exists $c > 0$ such that for all $\eta \in \mathcal{V}(\mathbb{R}^N)$

$$\begin{aligned}\langle A\eta,\eta\rangle &= \int_{\mathbb{R}^N}\gamma\left[T(U_0+\nabla\eta)-T(U_0)\right].\nabla\eta\\ &\geq c\underline{\gamma}(\|\nabla\eta\|^p_{L^p(\mathbb{R}^N)} + \|\nabla\eta\|^2_{L^2(\mathbb{R}^N)}).\end{aligned}$$

It follows from Proposition A.5 that

$$\lim_{\|\eta\|\to\infty}\frac{\langle A\eta,\eta\rangle}{\|\eta\|_{\mathcal{V}(\mathbb{R}^N)}} = +\infty,$$

whereby A is coercive in $\mathcal{V}(\mathbb{R}^N)$.

Therefore by virtue of the Minty-Browder theorem ([27], Theorem V-15), there exists a unique $H \in \mathcal{V}(\mathbb{R}^N)$ such that $AH = L$, which completes the proof of Proposition 3.7. □

3.4.5. *Step 4: asymptotic behavior of variations of the direct state.* For all $\varepsilon > 0$ small enough, set

$$H_\varepsilon(x) := \varepsilon\hat{H}(\varepsilon^{-1}x) \tag{3.22}$$

where $\hat{H} \in \mathcal{V}^w(\mathbb{R}^N)$ is for the time being an arbitrary element of the class $H \in \mathcal{V}(\mathbb{R}^N)$.

As $\inf_{x\in\Omega} w_p\left(\varepsilon^{-1}x\right) > 0$, it follows from $\hat{H} \in \mathcal{V}^w(\mathbb{R}^N)$ that $H_\varepsilon \in W^{1,p}(\Omega) \subset H^1(\Omega)$. Using the combined p- and 2-ellipticity property stated in condition (5), we obtain the following estimates.

Lemma 3.8. *It holds:*

$$\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \tag{3.23}$$

$$\|\nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \tag{3.24}$$

$$\|\nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla H_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N). \tag{3.25}$$

The proof is available in subsection 4.5 on page 28.

Further estimation of the variation of the direct state at scale ε requires to estimate the asymptotic behavior of function H at scale 1 in $\mathbb{R}^N$. To our best knowledge, no such result is available in the literature, e.g. [53, 54, 56, 60].

Let us first study the asymptotic decay of a radial function of $\mathcal{V}(\mathbb{R}^N)$. Let $\eta \in \mathcal{V}^w(\mathbb{R}^N)$ such that for some $C, \tau \in \mathbb{R}$ and $M > 0$, it holds

$$\eta(x) = C\,|x|^{-\tau}, \quad \forall x \in \mathbb{R}^N, |x| \geq M.$$

By definition of $\mathcal{V}^w(\mathbb{R}^N)$, it holds $w_p\eta \in L^p(\mathbb{R}^N)$ and $\nabla\eta \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. An easy calculation shows that these conditions imply

$$\tau > \frac{N}{2} - 1$$

and that, whatever the values of $N \geq 2$ and $p \in [2, \infty)$, the constraint $\nabla\eta \in L^2(\mathbb{R}^N)$ is always the active one with respect to exponent τ.

In the subsequent we shall prove that the asymptotic decay of function H is similar to that of a radial function of $\mathcal{V}(\mathbb{R}^N)$, at least in the prototype case, that is when $\omega = B(0,1)$ and $W = W_a$. Accordingly, relaxing the assumption about the shape of ω and assuming only that potential W belongs to the class of potentials defined by Assumption 3.1, we shall make the Assumption 3.13 hereafter about the asymptotic behavior of function H.

3.4.6. *Asymptotic behavior of function H.* Denote Q the operator such that $QH = 0$, that is

$$\langle Q\eta_1, \eta_2\rangle := \int_{\mathbb{R}^N} \gamma\left[T(U_0 + \nabla\eta_1) - T(U_0)\right].\nabla\eta_2 + (\gamma_1 - \gamma_0)\int_\omega T(U_0).\nabla\eta_2,$$
$$\forall \eta_1, \eta_2 \in \mathcal{V}(\mathbb{R}^N). \quad (3.26)$$

We assume $\omega = B(0,1)$ and $W = W_a$ for some $a > 0$. Assuming again the non trivial case $U_0 \neq 0$, set $e_1 = |U_0|^{-1} U_0$ and let $(e_1, e_2, ..., e_N)$ be an orthonormal basis of $\mathbb{R}^N$. Denote $(x_1, x_2, ..., x_N)$ the system of coordinates in this basis.

Denote $\mathbb{R}^N_+$ the half-space $\left\{x \in \mathbb{R}^N; U_0.\ x \geq 0\right\}$. Due to the symmetry of $\omega = B(0,1)$ with respect to the line $\mathbb{R}U_0$, it follows straightforwardly from the uniqueness stated in Proposition 3.7, that H is odd with respect to the first coordinate. Thus there exists an element $\tilde{H}$ of the class H such that

$$\tilde{H}(-x_1, x_2, ..., x_N) = -\tilde{H}(x_1, x_2, ..., x_N), \quad \forall (x_1, x_2, ..., x_N) \in \mathbb{R}^N.$$

In particular it holds

$$\tilde{H}(0, x_2, ..., x_N) = 0, \quad \forall (x_2, ..., x_N) \in \mathbb{R}^{N-1}.$$

Hence it suffices to study the asymptotic behavior of function $\tilde{H}$ in the half-space $\mathbb{R}^N_+$.

We denote

$$\bar{p} := 2 + \left(1 + \frac{a^2}{|U_0|^2}\right)\frac{N}{N-2}, \quad (3.27)$$

with the convention that $\bar{p} = +\infty$ if $N = 2$. The following results provide adequate supersolution and subsolution of function $\tilde{H}$, respectively.

Proposition 3.9. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. If $p \in [2, \bar{p})$, then there exists $\beta > N/2$ and a function $P \in \mathcal{V}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that*

$$\begin{cases} P(x) = k(U_0.x)\,|x|^{-\beta}, & \textit{if } |x| > 1, \\ P(x) = k(U_0.x), & \textit{if } |x| \leq 1, \end{cases} \quad (3.28)$$

$$\langle QP, \eta\rangle \geq 0, \qquad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \mathrm{spt}(\eta) \subset \mathbb{R}^N_+, \eta \geq 0 \ \textit{a.e.},$$

where

$$k := \frac{\gamma_0 - \gamma_1}{\gamma_1 + \gamma_0(\beta - 1)}. \quad (3.29)$$

The proof is available in subsection 4.6 on page 29.

Lemma 3.10. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. Denote 0 the null function in $\mathbb{R}^N$. Then*

$$\langle Q0, \eta\rangle \leq 0, \qquad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \mathrm{spt}(\eta) \subset \mathbb{R}^N_+, \eta \geq 0 \ a.e. \ .$$

The proof is easy following the steps taken for proving Proposition (3.9). Only is the transmission condition to be checked across $\partial\omega$, as obviously $Q0 = 0$ in ω and in $\mathbb{R}^N \setminus \bar{\omega}$.

Proposition 3.11. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$ and $p \in [2, \bar{p})$. Let P be the supersolution defined in Proposition (3.9). Then there exists an element $\tilde{H}$ of the class H such that*

$$0 \leq \tilde{H}(x) \leq P(x), \quad \text{for a.e. } x \in \mathbb{R}^N_+. \tag{3.30}$$

The proof is available in subsection 4.7 on page 32. Therefore we can now state

Theorem 3.12. *Assume $\omega = B(0,1)$, $\gamma_1 < \gamma_0$ and $W = W_a$ for some $a > 0$. Assume $p \in [2, \bar{p})$ where $\bar{p}$ is defined by equation (3.27).*
Then there exists an element $\tilde{H}$ of the class $H \in \mathcal{V}(\mathbb{R}^N)$ and $\tau > \frac{N}{2} - 1$ such that

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty. \tag{3.31}$$

Moreover it holds

$$H \in L^\infty(\mathbb{R}^N). \tag{3.32}$$

This completes our analysis of the asymptotic behavior of H in the case $\omega = B(0,1)$ and $W = W_a$. Accordingly we make the following assumption in the general case.

Assumption 3.13. We assume that

(1) there exists an element $\tilde{H}$ of the class $H \in \mathcal{V}(\mathbb{R}^N)$ and $\tau > \frac{N}{2} - 1$ such that

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty; \tag{3.33}$$

(2) and

$$H \in L^\infty(\mathbb{R}^N). \tag{3.34}$$

Lemma 3.14. *It holds $H \in \mathcal{H}(\mathbb{R}^N)$*

The proof is available in subsection 4.8 on page 33.
From now on, function H_ε is defined choosing $\hat{H} = \tilde{H}$ in (3.22), i.e.

$$H_\varepsilon(x) := \varepsilon \tilde{H}(\varepsilon^{-1}x), \quad \forall x \in \Omega. \tag{3.35}$$

Proposition 3.15. *It holds*

$$\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N), \tag{3.36}$$

$$\forall \alpha > 0, \ \forall r \in (0,1), \ \int_{\Omega \setminus B(0, \alpha\varepsilon^r)} |\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 = o(\varepsilon^N), \tag{3.37}$$

$$\int_\Omega |\nabla u_0 - U_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) = o(\varepsilon^N), \tag{3.38}$$

$$\forall \ p \in (4, \infty), \int_\Omega |\nabla u_0 - U_0| \, |\nabla h_\varepsilon|^{p-2} = o(\varepsilon^N), \tag{3.39}$$

$$\forall \ p \in (3, \infty), \int_\Omega |\nabla u_0 - U_0| \, |\nabla h_\varepsilon|^{p-1} = o(\varepsilon^N), \tag{3.40}$$

$$\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right) = o(\varepsilon^N), \tag{3.41}$$

$$\forall \ p \in (3, \infty), \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-2} = o(\varepsilon^N). \tag{3.42}$$

The proof is available in subsection 4.9 on page 33.

Proposition 3.16. *It holds*

$$\begin{aligned}
\|\nabla\tilde{u}_\varepsilon-\nabla h_\varepsilon\|^p_{L^p(\Omega)}+\|\nabla\tilde{u}_\varepsilon-\nabla h_\varepsilon\|^2_{L^2(\Omega)} &= o(\varepsilon^N), && (3.43)\\
\int_\Omega |\nabla\tilde{u}_\varepsilon-\nabla h_\varepsilon|\left(|\nabla\tilde{u}_\varepsilon|+|\nabla h_\varepsilon|\right) &= o(\varepsilon^N), && (3.44)\\
\forall\ p\in(3,\infty),\int_\Omega |\nabla\tilde{u}_\varepsilon-\nabla h_\varepsilon|\left(|\nabla\tilde{u}_\varepsilon|+|\nabla h_\varepsilon|\right)^{p-2} &= o(\varepsilon^N), && (3.45)\\
\forall\alpha>0,\ \forall\ r\in(0,1),\ \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla\tilde{u}_\varepsilon|^p+|\nabla\tilde{u}_\varepsilon|^2 &= o(\varepsilon^N). && (3.46)
\end{aligned}$$

The proof is available in subsection 4.10 on page 36.

Estimate (3.46) states that the energy of the variation outside a ball of radius $\alpha\varepsilon^r$, $r\in(0,1)$ can be neglected at first order in the asymptotic expansion. When $\varepsilon\to 0$, the radius of the ball $B(0,\alpha\varepsilon^r)$ goes to 0, but compared to the size of the perturbation subdomain ω_ε it goes to infinity. The radius $\alpha\varepsilon^r$ follows directly from the asymptotic behavior of function H.

3.5. **Variation of the adjoint state.** We define the adjoint state as solution of a linearized equation in the Hilbert space $\mathcal{H}=H^1_0(\Omega)$. Using the coercivity provided by condition (4) and the Lax-Milgram theorem in $\mathcal{H}$, one obtains that, for all $\varepsilon\geq 0$ small enough, there exists a unique $v_\varepsilon\in\mathcal{H}$ solution of variational problem

$$\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla v_\varepsilon.\nabla\eta=-\left\langle G,\eta\right\rangle,\quad \forall\eta\in\mathcal{H}. \tag{3.47}$$

3.5.1. *About the regularity of the unperturbed adjoint state.*

Lemma 3.17. *If the functional $\mathcal{J}_\varepsilon$ is the compliance (3.10), then it holds $v_0\in C^{1,\tilde{\beta}}(\overline{\Omega})$ for some $\tilde{\beta}>0$. In particular this implies $v_0\in L^\infty(\Omega)$, $\nabla v_0\in L^\infty(\Omega)$ and $v_0\in\mathcal{V}$.*

The proof is available in subsection 4.11 on page 39. Accordingly, in the general case we shall make the assumptions that $v_0\in L^\infty(\Omega)$ and $\nabla v_0\in L^\infty(\Omega)$. As by definition $v_0\in\mathcal{H}$, it follows that $v_0\in\mathcal{V}$.

3.5.2. *Step* 1*: variation $v_\varepsilon-v_0$.* Let $\tilde{v}_\varepsilon=v_\varepsilon-v_0$. After (3.47), one obtains

$$\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{v}_\varepsilon.\nabla\eta=-\left(\gamma_1-\gamma_0\right)\int_{\omega_\varepsilon} DT(\nabla u_0)\nabla v_0.\nabla\eta,\quad \forall\ \eta\in\mathcal{H}. \tag{3.48}$$

3.5.3. *Step* 2*: approximation of variation $\tilde{v}_\varepsilon$.* Applying the Lax-Milgram theorem, we approximate the variation $\tilde{v}_\varepsilon$ by the unique function $k_\varepsilon\in\mathcal{H}$ such that

$$\int_\Omega \gamma_\varepsilon DT(U_0)\nabla k_\varepsilon.\nabla\eta=-\left(\gamma_1-\gamma_0\right)\int_{\omega_\varepsilon} DT(U_0)V_0.\nabla\eta,\quad \forall\eta\in\mathcal{H}. \tag{3.49}$$

3.5.4. *Step* 3*: change to scale* 1*.* We look for a function K which may approximate the map $y\in\Omega/\varepsilon\mapsto\varepsilon^{-1}k_\varepsilon(\varepsilon y)$. The weighted quotient Hilbert space $\mathcal{H}(\mathbb{R}^N)$ is defined in section A.2. Applying the Lax-Milgram theorem in $\mathcal{H}(\mathbb{R}^N)$, one obtains

Lemma 3.18. *There exists a unique function $K\in\mathcal{H}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} \gamma DT(U_0)\nabla K.\nabla\eta=-\left(\gamma_1-\gamma_0\right)\int_\omega DT(U_0)V_0.\nabla\eta,\quad \forall\ \eta\in\mathcal{H}(\mathbb{R}^N). \tag{3.50}$$

3.5.5. *Step 4: asymptotic behavior of variations of the adjoint state.* For all $\varepsilon > 0$ small enough, let

$$K_\varepsilon : x \in \Omega \mapsto K_\varepsilon(x) := \varepsilon \hat{K}(\varepsilon^{-1}x) \tag{3.51}$$

where $\hat{K} \in \mathcal{H}^w(\mathbb{R}^N)$ is for the time being an arbitrary element of the class $K \in \mathcal{H}(\mathbb{R}^N)$. Making the change of scale backward, as $\inf_{x\in\Omega} w_2\left(\varepsilon^{-1}x\right) > 0$, it follows from $\hat{K} \in \mathcal{H}^w(\mathbb{R}^N)$ that $K_\varepsilon \in H^1(\Omega)$.

Lemma 3.19. *It holds*

$$\|\nabla \tilde{v}_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \tag{3.52}$$

$$\|\nabla k_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N), \tag{3.53}$$

$$\|\nabla K_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N). \tag{3.54}$$

The proof is available in subsection 4.12 on page 39.

Proposition 3.20. *There exists an element $\tilde{K}$ of the class $K \in \mathcal{H}(\mathbb{R}^N)$ such that*

$$\tilde{K}(y) = O\left(|y|^{1-N}\right) \quad as \quad |y| \to +\infty, \tag{3.55}$$

and

$$\nabla K(y) = O\left(|y|^{-N}\right) \quad as \quad |y| \to +\infty. \tag{3.56}$$

Moreover, it holds

$$K \in \mathcal{V}(\mathbb{R}^N). \tag{3.57}$$

The proof is available in subsection 4.13 on page 39.

From now on, function K_ε will be defined choosing $\hat{K} = \tilde{K}$ in (3.51).

Lemma 3.21. *It holds*

$$\|\nabla k_\varepsilon - \nabla K_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N), \tag{3.58}$$

and

$$\forall \alpha > 0, \forall r \in (0,1), \quad \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2 = o(\varepsilon^N). \tag{3.59}$$

The proof is available in subsection 4.14 on page 41.

Lemma 3.22. *It holds*

$$\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|^2_{L^2(\Omega)} = o(\varepsilon^N). \tag{3.60}$$

The proof is available in subsection 4.15 on page 41.

3.6. **Topological asymptotic expansion.** For simplicity we denote

$$j(\varepsilon) := \mathcal{J}_\varepsilon(u_\varepsilon), \quad \forall \varepsilon \geq 0 \text{ small enough.} \tag{3.61}$$

Expansion (3.7) reads

$$j(\varepsilon) - j(0) = \langle G, \tilde{u}_\varepsilon \rangle + \delta_2\, \varepsilon^N + R(\varepsilon).$$

- In the first case (3.8), after estimate (3.23), it holds
$$R(\varepsilon) = o\left(\|u_\varepsilon - u_0\|^2_{\mathcal{H}}\right) = o\left(\varepsilon^N\right).$$
- In the second case (3.9), after estimate (3.46), it holds
$$R(\varepsilon) = O\left(\int_{\Omega\setminus B(0,\tilde{\alpha}\varepsilon^{\tilde{r}})} |\nabla \tilde{u}_\varepsilon|^p + |\nabla \tilde{u}_\varepsilon|^2\right) = o\left(\varepsilon^N\right).$$

Then plugging test function $\eta = \tilde{u}_\varepsilon \in \mathcal{V} \subset \mathcal{H}$ in variational form (3.47), one obtains

$$\begin{aligned} j(\varepsilon) - j(0) &= -\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{u}_\varepsilon.\nabla v_\varepsilon + \delta_2\ \varepsilon^N + o(\varepsilon^N) \\ &= -\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{u}_\varepsilon.\nabla v_0 - \int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{u}_\varepsilon.\nabla\tilde{v}_\varepsilon + \delta_2\ \varepsilon^N + o(\varepsilon^N). \end{aligned} \quad (3.62)$$

Plugging now $\eta = \tilde{u}_\varepsilon \in \mathcal{V} \subset \mathcal{H}$ in variational form (3.48) yields

$$\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla\tilde{v}_\varepsilon.\nabla\tilde{u}_\varepsilon + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon = 0. \quad (3.63)$$

Since we have assumed that $v_0 \in \mathcal{V}$ (see Lemma 3.17), we can plug $\eta = v_0$ in variational form (3.17) and obtain

$$\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla\tilde{u}_\varepsilon) - T(\nabla u_0)\right].\nabla v_0 + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_0 = 0. \quad (3.64)$$

Summing equalities (3.62), (3.63) and (3.64), it follows

$$j(\varepsilon) - j(0) = j_1(\varepsilon) + j_2(\varepsilon) + \delta_2\ \varepsilon^N + o(\varepsilon^N) \quad (3.65)$$

with

$$j_1(\varepsilon) := (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_\varepsilon \quad (3.66)$$

and

$$\begin{aligned} j_2(\varepsilon) := &\int_\Omega \gamma_\varepsilon \left[T(\nabla\tilde{u}_\varepsilon + \nabla u_0) - T(\nabla u_0) - DT(\nabla u_0)(\nabla\tilde{u}_\varepsilon)\right].\nabla v_0 \\ &+ (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} \left[DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - T(\nabla u_0).\nabla\tilde{v}_\varepsilon\right] \\ = &\int_\Omega \gamma_\varepsilon S_{\nabla u_0}(\nabla\tilde{u}_\varepsilon).\nabla v_0 + (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} \left[DT(\nabla u_0)\nabla v_0.\nabla\tilde{u}_\varepsilon - T(\nabla u_0).\nabla\tilde{v}_\varepsilon\right]. \end{aligned} \quad (3.67)$$

We shall see hereafter that $j_2(\varepsilon)$ accounts for the contribution of the nonlinear behavior of T to $j(\varepsilon)$, while $j_1(\varepsilon)$ provides the variation of $j(\varepsilon)$ caused by the 'affine component' of T.

3.6.1. *Expansion of linear term $j_1(\varepsilon)$.* Following the approximation steps 2 and 3, we define

$$\tilde{j}_1(\varepsilon) \quad := \quad (\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(U_0).(V_0 + \nabla k_\varepsilon), \quad (3.68)$$

and

$$\begin{aligned} J_1 \quad &:= \quad (\gamma_1 - \gamma_0)\int_\omega T(U_0).(V_0 + \nabla K) \\ &= \quad (\gamma_1 - \gamma_0)\ T(U_0).\left[|\omega|\,V_0 + \int_{\partial\omega} Kn\right], \end{aligned} \quad (3.69)$$

the last equality stemming from Green's formula, where n is the outward unit normal to $\partial\omega$.

Regarding the calculation of the latter integral, it follows from the linearity of equation (3.50) that the mapping

$$V_0 \mapsto (\gamma_1 - \gamma_0)\left[|\omega|\,V_0 + \int_{\partial\omega} Kn\right]$$

is linear $\mathbb{R}^N \to \mathbb{R}^N$. It only depends on the set ω, on the definite positive matrix $DT(U_0)$ and on the ratio γ_1/γ_0. Hence there exists a second order tensor, called *polarization tensor*,

$$\mathcal{P} = \mathcal{P}(\omega, DT(U_0), \gamma_1/\gamma_0),$$

such that

$$(\gamma_1-\gamma_0)\left[|\omega|\,V_0+\int_{\partial\omega}Kn\right] \;=\; \mathcal{P}V_0. \tag{3.70}$$

Eventually we arrive at

$$J_1 = T(U_0).\,(\mathcal{P}V_0) = T(U_0)^T\mathcal{P}V_0. \tag{3.71}$$

Lemma 3.23. *For all $\varepsilon \geq 0$ small enough it holds*

$$\tilde{j}_1(\varepsilon) - \varepsilon^N J_1 = o(\varepsilon^N). \tag{3.72}$$

The proof is available in subsection 4.16 on page 43.

Lemma 3.24. *For all $\varepsilon \geq 0$ small enough it holds*

$$j_1(\varepsilon) - \tilde{j}_1(\varepsilon) = o(\varepsilon^N). \tag{3.73}$$

The proof is available in subsection 4.17 on page 44.

Summing estimates (3.72) and (3.73) we derive the following.

Proposition 3.25. *It holds:*

$$j_1(\varepsilon) = \varepsilon^N T(U_0)^T\mathcal{P}V_0 + \; o\left(\varepsilon^N\right). \tag{3.74}$$

3.6.2. *Expansion of nonlinear term $j_2(\varepsilon)$.* The term $j_2(\varepsilon)$ can be approximated with the help of approximation steps 2 and 3 as follows. Since $\nabla h_\varepsilon \in L^p(\Omega)$ and $\nabla k_\varepsilon \in L^2(\Omega)$, in view of the growth condition (3.2) one can define

$$\tilde{j}_2(\varepsilon) := \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla h_\varepsilon).V_0 + (\gamma_1-\gamma_0)\int_{\omega_\varepsilon}\left[DT(U_0)V_0.\nabla h_\varepsilon - T(U_0).\nabla k_\varepsilon\right]. \tag{3.75}$$

Similarly since $\nabla H \in L^p(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and $\nabla K \in L^2(\mathbb{R}^N)$, one can define

$$J_2 := \int_{\mathbb{R}^N}\gamma S_{U_0}(\nabla H).V_0 + (\gamma_1-\gamma_0)\int_\omega\left[DT(U_0)V_0.\nabla H - T(U_0).\nabla K\right]. \tag{3.76}$$

In addition after Propositions 3.14 and 3.20, it holds $H \in \mathcal{H}(\mathbb{R}^N)$ and $K \in \mathcal{V}(\mathbb{R}^N)$. Plugging test function K into variational form (3.19) defining H and plugging test function H into variational form(3.50) defining K, one obtains

$$J_2 \;=\; \int_{\mathbb{R}^N}\gamma S_{U_0}(\nabla H).(V_0+\nabla K). \tag{3.77}$$

Lemma 3.26. *For $\varepsilon \geq 0$ small enough it holds*

$$\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 = o(\varepsilon^N). \tag{3.78}$$

The proof is available in subsection 4.18 on page 44.

Lemma 3.27. *It holds*

$$\int_\Omega |\nabla v_0 - V_0|\left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) \;=\; o(\varepsilon^N), \tag{3.79}$$

$$\forall\; p\in(3,\infty), \int_\Omega |\nabla v_0 - V_0|\,|\nabla h_\varepsilon|^{p-1} \;=\; o(\varepsilon^N). \tag{3.80}$$

The proof is available in subsection 4.19 on page 45.

Lemma 3.28. *For $\varepsilon \geq 0$ small enough it holds*

$$j_2(\varepsilon) - \tilde{j}_2(\varepsilon) = o(\varepsilon^N). \tag{3.81}$$

The proof is available in subsection 4.20 on page 45.

Eventually summing estimates (3.78) and (3.81) yields the following.

Proposition 3.29.

$$j_2(\varepsilon) = \varepsilon^N\left(\int_{\mathbb{R}^N}\gamma S_{U_0}(\nabla H).\,(V_0+\nabla K)\right) + o\left(\varepsilon^N\right). \tag{3.82}$$

3.6.3. *Topological asymptotic expansion.* Lastly, according to (3.65), summing the estimate of $j_1(\varepsilon)$ given by (3.74) and the estimate of $j_2(\varepsilon)$ given by (3.82), one obtains the topological asymptotic expansion claimed in Theorem 3.5.

4. Proofs

4.1. **Proof of Proposition 3.2.** Let $\varphi, \psi \in \mathbb{R}^N$. It is easy to check the following algebraic identity by expanding both sides

$$\begin{aligned} R(\varphi,\psi) &:= \left[(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}}(\varphi+\psi) - (a^2+|\varphi|^2)^{\frac{p-2}{2}}\varphi\right].\psi \\ &= \frac{(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}} - (a^2+|\varphi|^2)^{\frac{p-2}{2}}}{2}\left(|\varphi+\psi|^2-|\varphi|^2\right) \\ &\qquad\qquad + \frac{(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}} + (a^2+|\varphi|^2)^{\frac{p-2}{2}}}{2}\,|\psi|^2 . \end{aligned}$$

By monotonicity of the function $x \in \mathbb{R}_+ \mapsto (a^2+x)^{\frac{p-2}{2}}$, the first term on the right-hand side is always non negative. Hence

$$R(\varphi,\psi) \geq \frac{(a^2+|\varphi+\psi|^2)^{\frac{p-2}{2}} + (a^2+|\varphi|^2)^{\frac{p-2}{2}}}{2}\,|\psi|^2 . \tag{4.1}$$

It follows immediately from (4.1) that

$$R(\varphi,\psi) \geq c_2\,|\psi|^2 ,$$

where $c_2 := a^{p-2} > 0$ does not depend on φ, ψ.

Let us now prove the p-coercivity. As the case $\psi = 0$ is trivial, we assume $\psi \neq 0$ and we decompose φ as follows

$$\varphi = \xi + s\psi, \text{ with } \xi \in \mathbb{R}^N, s \in \mathbb{R} \text{ and } \xi.\psi = 0.$$

Let $b \in \mathbb{R}_+$ such that $b^2 = a^2 + |\xi|^2$. The Pythagorean theorem yields

$$R(\varphi,\psi) = \left[\left(b^2+(s+1)^2|\psi|^2\right)^{\frac{p-2}{2}}(s+1) - \left(b^2+s^2|\psi|^2\right)^{\frac{p-2}{2}}s\right]|\psi|^2.$$

Let

$$d := \frac{b}{|\psi|} \quad \text{ and } \quad Q(\varphi,\psi) := \frac{R(\varphi,\psi)}{|\psi|^p}.$$

Thus

$$Q(\varphi,\psi) = \left(d^2+(s+1)^2\right)^{\frac{p-2}{2}}(s+1) - \left(d^2+s^2\right)^{\frac{p-2}{2}}s. \tag{4.2}$$

We distinguish between several cases.

(1) If $s \geq 0$. Due to the monotonicity of the map $x \in \mathbb{R}_+ \mapsto (d^2+x^2)^{\frac{p-2}{2}}$, it holds

$$\left[\left(d^2+(s+1)^2\right)^{\frac{p-2}{2}} - \left(d^2+s^2\right)^{\frac{p-2}{2}}\right]s \geq 0$$

and

$$\left(d^2+(s+1)^2\right)^{\frac{p-2}{2}} \geq \left(d^2+1\right)^{\frac{p-2}{2}} \geq 1.$$

It thus follows from formula (4.2) that $Q(\varphi,\psi) \geq 1$.

(2) If $s < 0$. Let $t := |s| = -s$. Rewrite (4.2) as follows

$$Q(\varphi,\psi) = \left(d^2+(1-t)^2\right)^{\frac{p-2}{2}}(1-t) + \left(d^2+t^2\right)^{\frac{p-2}{2}}t. \tag{4.3}$$

We distinguish again two cases.

(a) If $t \leq 1$. Due to the monotonicity of the maps $x \in \mathbb{R}_+ \mapsto x^{\frac{p-2}{2}}$ and $x \in \mathbb{R}_+ \mapsto x^{p-1}$ and since

$$\max(1-x, x) \geq \frac{1}{2}, \quad \forall x \in (0,1],$$

one obtains from (4.3) that

$$Q(\varphi,\psi) \geq (1-t)^{p-2}(1-t) + t^{p-2}t = (1-t)^{p-1} + t^{p-1} \geq 2^{1-p}.$$

(b) If $t > 1$. As $0 < t-1 < t$, by monotonicity, it holds

$$-\left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} + \left(b^2 + t^2\right)^{\frac{p-2}{2}} \geq 0.$$

From equation (4.3), one thus obtains

$$\begin{aligned} Q(\varphi,\psi) &= \left[-\left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} + \left(d^2 + t^2\right)^{\frac{p-2}{2}}\right] t + \left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} \\ &\geq \left[-\left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} + \left(d^2 + t^2\right)^{\frac{p-2}{2}}\right] + \left(d^2 + (1-t)^2\right)^{\frac{p-2}{2}} \\ &= \left(d^2 + t^2\right)^{\frac{p-2}{2}} \geq \left(d^2 + 1\right)^{\frac{p-2}{2}} \geq 1. \end{aligned}$$

Let $c_p = \min(1, 2^{1-p}) = 2^{1-p}$. We have thus proved that

$$R(\varphi,\psi) \geq c_p \, |\psi|^p, \quad \forall\ \varphi,\psi \in \mathbb{R}^N.$$

Lastly choosing $c = \frac{1}{2}\min(c_2, c_p)$ completes the proof of Proposition 3.2.

4.2. **Proof of Proposition 3.3.** We shall prove that, for all $a > 0$, the potential W_a satisfies Assumption 3.1.

(1) The two maps $\varphi \in \mathbb{R}^N \mapsto |\varphi|^2$ and $t \in \mathbb{R}_+ \mapsto \frac{1}{p}(a^2 + t)^{\frac{p}{2}}$ are C^∞. It follows that the composite function W_a is also C^∞.

(2) Regarding condition (2), it is obvious that the lower bound holds with $a_0 := \frac{1}{p}$. Then, for all $\varphi \in \mathbb{R}^N$, we have by convexity that

$$W_a(\varphi) = \frac{1}{p}\left(a^2 + |\varphi|^2\right)^{\frac{p}{2}} \leq \frac{1}{p} 2^{\frac{p-2}{2}} \left(a^p + |\varphi|^p\right).$$

Hence the upper bound of condition (2) holds choosing $b_0 := \frac{1}{p} 2^{\frac{p-2}{2}} \max(a^p, 1)$.

(3) About condition (3), it first holds

$$T_a(\varphi) = \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} \varphi \quad \forall \varphi \in \mathbb{R}^N.$$

Thus

$$\begin{cases} |T_a(\varphi)| \leq 2^{\frac{p-2}{2}} a^{p-2} \, |\varphi| & \text{if } |\varphi| \leq a, \\ |T_a(\varphi)| \leq 2^{\frac{p-2}{2}} \, |\varphi|^{p-1} & \text{if } |\varphi| > a. \end{cases}$$

Hence inequality in condition (3) holds choosing $a_1 := 2^{\frac{p-2}{2}} \max(a^{p-2}, 1)$.

(4) For all $\varphi, \psi \in \mathbb{R}^N$ it holds

$$DT_a(\varphi)\psi = (p-2)\left(a^2 + |\varphi|^2\right)^{\frac{p-4}{2}} (\varphi.\psi)\varphi + \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} \psi.$$

Thus

$$\begin{aligned} DT_a(\varphi)\psi.\psi &= (p-2)\left(a^2 + |\varphi|^2\right)^{\frac{p-4}{2}} (\varphi.\psi)^2 + \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} |\psi|^2 \\ &\geq \left(a^2 + |\varphi|^2\right)^{\frac{p-2}{2}} |\psi|^2. \end{aligned}$$

Hence the lower bound in condition (4) holds choosing $c := \min(1, a^{p-2}) > 0$.

Moreover the Cauchy-Schwarz inequality yields

$$\begin{aligned} DT_a(\varphi)\psi.\psi &\leq \left[(p-2)\left(a^2+|\varphi|^2\right)^{\frac{p-4}{2}}|\varphi|^2+\left(a^2+|\varphi|^2\right)^{\frac{p-2}{2}}\right]|\psi|^2 \\ &\leq (p-1)\left(a^2+|\varphi|^2\right)^{\frac{p-2}{2}}|\psi|^2 . \end{aligned}$$

Hence the upper bound in condition (4) holds choosing $C := (p-1)\max(1, a^{p-2})$.

(5) Condition (5) follows immediately from Proposition 3.2.

(6) Regarding condition (6), let $\varphi, \psi \in \mathbb{R}^N$. Let $g : t \in (0,1) \mapsto T_a(\varphi + t\psi)$. The equality

$$g(1) - g(0) = \int_0^1 g'(t)\mathrm{d}t$$

may be expanded into

$$\begin{aligned} T_a(\varphi+\psi) - T_a(\varphi) &= (p-2)\int_0^1 \left(a^2+|\varphi+t\psi|^2\right)^{\frac{p-4}{2}}((\varphi+t\psi).\psi)(\varphi+t\psi)\ \mathrm{d}t \\ &\quad + \int_0^1 \left(a^2+|\varphi+t\psi|^2\right)^{\frac{p-2}{2}}\psi\ \mathrm{d}t. \end{aligned}$$

Thus

$$\begin{aligned} |T_a(\varphi+\psi) - T_a(\varphi)| &\leq (p-2)\int_0^1 \left(a^2+|\varphi+t\psi|^2\right)^{\frac{p-4}{2}}|\varphi+t\psi|^2|\psi|\ \mathrm{d}t \\ &\quad + \int_0^1 \left(a^2+|\varphi+t\psi|^2\right)^{\frac{p-2}{2}}|\psi|\ \mathrm{d}t \\ &\leq (p-1)|\psi|\int_0^1 \left(a^2+|\varphi+t\psi|^2\right)^{\frac{p-2}{2}}\ \mathrm{d}t \\ &\leq (p-1)|\psi|\left(a^2+2|\varphi|^2+2|\psi|^2\right)^{\frac{p-2}{2}} . \end{aligned} \tag{4.4}$$

Moreover

$$\begin{cases} \left(a^2+2|\varphi|^2+2|\psi|^2\right)^{\frac{p-2}{2}} \leq 2^{\frac{p-2}{2}}\left(a^2+2|\varphi|^2\right)^{\frac{p-2}{2}} & \text{if } 2|\psi|^2 \leq a^2+2|\varphi|^2, \\ \left(a^2+2|\varphi|^2+2|\psi|^2\right)^{\frac{p-2}{2}} \leq 2^{p-2}|\psi|^{p-2} & \text{if } 2|\psi|^2 > a^2+2|\varphi|^2. \end{cases}$$

Hence inequality (4.4) entails

$$\begin{aligned} |T_a(\varphi+\psi) - T_a(\varphi)| &\leq (p-1)|\psi|\left[2^{\frac{p-2}{2}}\left(a^2+2|\varphi|^2\right)^{\frac{p-2}{2}} + 2^{p-2}|\psi|^{p-2}\right] \\ &\leq C|\psi|\left[1+|\varphi|^{p-2}+|\psi|^{p-2}\right], \end{aligned}$$

with $C := (p-1)2^{\frac{p-2}{2}}\max(1, 2^{\frac{p-4}{2}})\max(a^{p-2}, 2^{\frac{p-2}{2}})$. This proves condition (6).

(7) Regarding condition (7), let $M > 0$ and let $\varphi \in B(0, M)$. For all $\psi_1, \psi_2 \in \mathbb{R}^N$, the Taylor formula reads

$$\begin{aligned} S_\varphi(\psi_2) - S_\varphi(\psi_1) &= T_a(\varphi+\psi_2) - T_a(\varphi+\psi_1) - DT_a(\varphi)(\psi_2-\psi_1) \\ &= \int_0^1 \left[DT_a(\varphi+\psi_1+t(\psi_2-\psi_1)) - DT_a(\varphi)\right](\psi_2-\psi_1))\,dt \\ &= \int_0^1\int_0^1 D^2T_a\left(\varphi+s\left[(1-t)\psi_1+t\psi_2\right]\right)\left((1-t)\psi_1+t\psi_2\right)(\psi_2-\psi_1)\,ds\,dt. \end{aligned} \tag{4.5}$$

For all $\xi_1,\xi_2,\xi_3\in\mathbb{R}^N$ one obtains by differentiation

$$DT_a(\xi_1)(\xi_2) \;=\; (p-2)\left(a^2+|\xi_1|^2\right)^{\frac{p-4}{2}}(\xi_1.\xi_2)\xi_1+\left(a^2+|\xi_1|^2\right)^{\frac{p-2}{2}}\xi_2.$$

Next one gets

$$\begin{aligned}D^2T_a(\xi_1)(\xi_2,\xi_3)&=(p-2)(p-4)\left(a^2+|\xi_1|^2\right)^{\frac{p-6}{2}}(\xi_1.\xi_2)(\xi_1.\xi_3)\xi_1\\&\quad+(p-2)\left(a^2+|\xi_1|^2\right)^{\frac{p-4}{2}}\left[(\xi_2.\xi_3)\xi_1+(\xi_1.\xi_2)\xi_3+(\xi_1.\xi_3)\xi_2\right].\end{aligned}\tag{4.6}$$

After Cauchy-Schwarz's inequality it follows

$$\left|D^2T_a(\xi_1)(\xi_2,\xi_3)\right| \;\le\; C(p)\left(a^2+|\xi_1|^2\right)^{\frac{p-3}{2}}|\xi_2|\,|\xi_3|\,.$$

where $C(p)=(p-2)(|p-4|+3)$.

- If $p\in[2,3]$ then

$$\left|D^2T_a(\xi_1)(\xi_2,\xi_3)\right|\le C(p)a^{p-3}\,|\xi_2|\,|\xi_3|\,.$$

Hence (4.5) entails

$$|S_\varphi(\psi_2)-S_\varphi(\psi_1)|\le C(p)a^{p-3}\,|\psi_2-\psi_1|\left(|\psi_1|+|\psi_2|\right).$$

Therefore condition (7) holds with $c_0=C(p)a^{p-3}$ and $c_{p-3}=0$.

- If $p\in(3,\infty)$, for all $s,t\in(0,1)$ it holds

$$\begin{aligned}\left(a^2+|\varphi+s\left[(1-t)\psi_1+t\psi_2\right]|^2\right)^{\frac{p-3}{2}}&\le\left(a^2+2\,|\varphi|^2+2\left(|\psi_1|+|\psi_2|\right)^2\right)^{\frac{p-3}{2}}\\&\le 2^{\frac{p-3}{2}}\left(a^2+2M^2\right)^{\frac{p-3}{2}}+2^{p-3}\left(|\psi_1|+|\psi_2|\right)^{p-3}.\end{aligned}$$

Therefore (4.5) yields condition (7) with $c_0=2^{\frac{p-3}{2}}C(p)\left(a^2+2M^2\right)^{\frac{p-3}{2}}$ and $c_{p-3}=2^{p-3}C(p)$.

(8) Regarding condition (8), we set for clarity $Z_\psi(\varphi):=S_\varphi(\psi)$. For a given $\psi\in\mathbb{R}^N$, the map $\varphi\mapsto Z_\psi(\varphi)$ is C^∞. According to the Taylor formula with integral remainder, for all $\varphi,\psi,\xi\in\mathbb{R}^N$ it holds

$$\begin{aligned}DZ_\psi(\varphi)(\xi)&=DT_a(\varphi+\psi)(\xi)-DT_a(\varphi)(\xi)-D^2T_a(\varphi)(\psi,\xi)\\&=\int_0^1(1-s)D^3T_a(\varphi+s\psi)(\psi,\xi,\psi)\,ds.\end{aligned}$$

Let $M>0$ and let $\varphi_1,\varphi_2\in B(0,M)$. For all $\psi\in\mathbb{R}^N$ it thus holds

$$\begin{aligned}Z_\psi(\varphi_2)-Z_\psi(\varphi_1)&=\int_0^1 DZ_\psi(\varphi_1+t(\varphi_2-\varphi_1))(\varphi_2-\varphi_1)\,dt\\&=\int_0^1\int_0^1(1-s)D^3T_a(\varphi_1+t(\varphi_2-\varphi_1)+s\psi)(\psi,\varphi_2-\varphi_1,\psi)\,ds\,dt.\end{aligned}\tag{4.7}$$

Differentiating (4.6), for all $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^N$ one obtains

$$\begin{aligned} D^3T_a(\xi_1)(\xi_2,\xi_3,\xi_2) &= (p-2)(p-4)(p-6)\left(a^2+|\xi_1|^2\right)^{\frac{p-8}{2}}(\xi_1.\xi_2)^2(\xi_1.\xi_3)\xi_1 \\ &+ (p-2)(p-4)\left(a^2+|\xi_1|^2\right)^{\frac{p-6}{2}}\left[|\xi_2|^2(\xi_1.\xi_3)\xi_1+(\xi_1.\xi_2)(\xi_2.\xi_3)\xi_1+(\xi_1.\xi_2)(\xi_1.\xi_3)\xi_2\right] \\ &+ (p-2)(p-4)\left(a^2+|\xi_1|^2\right)^{\frac{p-6}{2}}(\xi_1.\xi_2)\left[(\xi_2.\xi_3)\xi_1+(\xi_1.\xi_2)\xi_3+(\xi_1.\xi_3)\xi_2\right] \\ &+ (p-2)\left(a^2+|\xi_1|^2\right)^{\frac{p-4}{2}}\left[|\xi_2|^2\xi_3+2(\xi_2.\xi_3)\xi_2\right]. \end{aligned}$$

Thus

$$\left|D^3T_a(\xi_1)(\xi_2,\xi_3,\xi_2)\right| \le C(p)\left(a^2+|\xi_1|^2\right)^{\frac{p-4}{2}}|\xi_2|^2|\xi_3|.$$

where $C(p) = (p-2)[3+|p-4|(6+|p-6|)]$.

- If $p \in [2,4]$, it holds

$$\left|D^3T_a(\xi_1)(\xi_2,\xi_2,\xi_3)\right| \le C(p)a^{p-4}|\xi_2|^2|\xi_3|.$$

  Hence it follows from (4.7)

$$|S_{\varphi_2}(\psi)-S_{\varphi_1}(\psi)| \le \frac{1}{2}C(p)a^{p-4}|\varphi_2-\varphi_1||\psi|^2.$$

  Therefore condition (8) holds with $d_0 = C(p)a^{p-4}/2$ and $d_{p-4} = 0$.
- If $p > 4$, for all $t, s \in (0,1)$ it holds

$$\begin{aligned} \left(a^2+|\varphi_1+t(\varphi_2-\varphi_1)+s\psi|^2\right)^{\frac{p-4}{2}} &\le \left(a^2+2M^2+2|\psi|^2\right)^{\frac{p-4}{2}} \\ &\le 2^{\frac{p-4}{2}}\left(a^2+2M^2\right)^{\frac{p-4}{2}}+2^{p-4}|\psi|^{p-4}. \end{aligned}$$

  Hence (4.7) yields

$$|S_{\varphi_2}(\psi)-S_{\varphi_2}(\psi)| \le |\varphi_2-\varphi_1|\left(d_0|\psi|^2+d_{p-4}|\psi|^{p-2}\right)$$

  with $d_0(M,p) = 2^{\frac{p-6}{2}}C(p)\left(a^2+2M^2\right)^{\frac{p-4}{2}}$ and $d_{p-4}(p) = 2^{p-5}C(p)$.

This completes the proof of condition (8).

4.3. **Proof of Lemma 3.4.** It follows from the upper-bound of condition (2) and from $f \in C^{0,\alpha}(\overline{\Omega}) \subset L^q(\Omega)$ that the functional $\mathcal{W}_\varepsilon$ is well defined in $\mathcal{V}$.

(1) Applying the Lebesgue dominated convergence theorem and using the assumptions, one proves in a standard way that functional $\mathcal{W}_\varepsilon$ is continuous and Gâteaux-differentiable in $\mathcal{V}$, with

$$D\mathcal{W}_\varepsilon(u)(\eta) = \int_\Omega \left[\gamma_\varepsilon T(\nabla u).\nabla\eta - f\eta\right], \quad \forall u,\eta \in \mathcal{V}. \tag{4.8}$$

See e.g. [18], proof of Thm. 6.6.1. Note that, according to condition (3), $\nabla u \in L^p(\Omega)$ implies that $T(\nabla u) \in L^q(\Omega)$. Hence the integral in (4.8) is well defined.

(2) The strict convexity of functional $\mathcal{W}_\varepsilon$ follows immediately from that of W.

(3) After Poincaré inequality in $\mathcal{V}$ and after condition (2), it holds

$$\mathcal{W}_\varepsilon(u) \ge \underline{\gamma}a_0|u|_{\mathcal{V}}^p - C\|f\|_{L^q(\Omega)}|u|_{\mathcal{V}}, \quad \forall u \in \mathcal{V},$$

which entails the coercivity of $\mathcal{W}_\varepsilon$ in $\mathcal{V}$.

Therefore (see e.g. [18], Theorem 3.3.4.) the minimization of $\mathcal{W}_\varepsilon$ in $\mathcal{V}$ admits a unique solution. This solution is equivalently defined by the first order condition $D\mathcal{W}_\varepsilon(u_\varepsilon) = 0$ which is the claimed Euler-Lagrange equation.

4.4. **Proof of Lemma 3.6.** The unperturbed direct state $u_0 \in \mathcal{V}$ is weak solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\gamma_0 T(\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

According to condition (1) it holds $T \in C^{1,\alpha}(\mathbb{R}^N)$ and by assumption $f \in C^{0,\alpha}(\overline{\Omega})$. Moreover referring to [46], structure conditions (3.46) p.181 hold by virtue of condition (4). Hence it follows from [46] Theorem 3.20 that $u_0 \in C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$.

4.5. **Proof of Lemma 3.8.**

(1) Plugging $\eta = \tilde{u}_\varepsilon \in \mathcal{V}$ the variational form (3.17) yields

$$\int_\Omega \gamma_\varepsilon \left[T(\nabla\tilde{u}_\varepsilon + \nabla u_0) - T(\nabla u_0)\right].\nabla\tilde{u}_\varepsilon \;=\; -\,(\gamma_1 - \gamma_0)\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\tilde{u}_\varepsilon. \tag{4.9}$$

It follows from condition (5) that there exists $c > 0$ such that

$$\underline{\gamma}\; c(\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)}) \le \int_\Omega \gamma_\varepsilon \left[T(\nabla\tilde{u}_\varepsilon + \nabla u_0) - T(\nabla u_0)\right].\nabla\tilde{u}_\varepsilon. \tag{4.10}$$

In addition $\nabla u_0 \in L^\infty(\bar{\Omega})$ by Lemma 3.6 and T is continuous. Thus let

$$M := \sup\left\{|T(\psi)|\,;\,|\psi| \le \|\nabla u_0\|_{L^\infty(\Omega)}\right\} < \infty.$$

- According to Hölder's inequality it holds

$$\left|\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\tilde{u}_\varepsilon\right| \le M\,|\omega|^{\frac{1}{q}}\,\varepsilon^{\frac{N}{q}}\,\|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)}\,.$$

  Therefore equations (4.9) and (4.10) imply

$$\underline{\gamma}\; c\,\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} \le |\gamma_1 - \gamma_0|\,M\,|\omega|^{\frac{1}{q}}\,\varepsilon^{\frac{N}{q}}\,\|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)}\,.$$

  Dividing both sides by $\|\nabla\tilde{u}_\varepsilon\|_{L^p(\Omega)}$ and powering the inequality to the power of q entails

$$\|\nabla\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} = O(\varepsilon^N).$$

- Similarly applying Cauchy-Schwarz's inequality, it holds

$$\left|\int_{\omega_\varepsilon} T(\nabla u_0).\nabla\tilde{u}_\varepsilon\right| \le M\,|\omega|^{\frac{1}{2}}\,\varepsilon^{\frac{N}{2}}\,\|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)}\,.$$

  Hence one obtains from (4.9) and (4.10) that

$$\|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} = \; O(\varepsilon^N),$$

  which completes the proof of (3.23).

(2) The proof of estimate (3.24) is similar to the one of (3.23), starting from variational form (3.18).

(3) Lastly, since $H \in \mathcal{V}(\mathbb{R}^N)$, by definition it holds $\nabla H \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus making a change of scale yields

$$\|\nabla H_\varepsilon\|^p_{L^p(\Omega)} = \int_\Omega |\nabla H_\varepsilon|^p = \varepsilon^N \int_{\Omega/\varepsilon} |\nabla H|^p \le \varepsilon^N\;\|\nabla H\|^p_{L^p(\mathbb{R}^N)} = O(\varepsilon^N).$$

Similarly one obtains

$$\|\nabla H_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N),$$

which completes the proof of estimate (3.25).

Remark 4.1. By convexity, it follows immediately from estimates (3.23), (3.24) and (3.25) that

$$\begin{aligned} \|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N), && (4.11)\\ \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N). && (4.12) \end{aligned}$$

Moreover $\tilde{u}_\varepsilon, h_\varepsilon \in \mathcal{V} \subset \mathcal{H}$. Thus according to Poincaré inequalities in $\mathcal{V}$ and in $\mathcal{H}$, inequalities (3.23) and (3.24) imply

$$\begin{aligned} \|\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} + \|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N), && (4.13)\\ \|h_\varepsilon\|^p_{L^p(\Omega)} + \|h_\varepsilon\|^2_{L^2(\Omega)} &= O(\varepsilon^N). && (4.14) \end{aligned}$$

By convexity again it follows from (4.13) and (4.14) that

$$\|\tilde{u}_\varepsilon - h_\varepsilon\|^p_{L^p(\Omega)} + \|\tilde{u}_\varepsilon - h_\varepsilon\|^2_{L^2(\Omega)} = O(\varepsilon^N). \tag{4.15}$$

4.6. **Proof of Proposition 3.9.** Consider the function P defined by (3.28) for some $\beta \in (N/2, N)$. It is easy to check that $P \in \mathcal{V}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Recall we denote

$$\mathbb{R}^N_+ = \left\{x \in \mathbb{R}^N;\ U_0.\ x \geq 0\right\}.$$

We shall prove that P is a supersolution of operator Q in the half-space $\mathbb{R}^N_+$ for some appropriately chosen $\beta \in (N/2, N)$. We shall need the following elementary inequalities: for all $\beta > N/2$, it holds

$$\begin{aligned} 1 + k(1-\beta) &> 0, && (4.16)\\ -2 + k(\beta - 2) &< 0. && (4.17) \end{aligned}$$

According to the Green formula, one can split operator Q into the sum of three operators Q_{int}, Q_{trans} and Q_{ext}, with supports respectively in $\bar{\omega}$, on $\partial\omega$ and in $\mathbb{R}^N \setminus \omega$, as follows:

$$\begin{aligned} \langle QP, \eta\rangle &:= \int_{\mathbb{R}^N} \gamma\left[T(U_0 + \nabla P) - T(U_0)\right].\nabla\eta + (\gamma_1 - \gamma_0)\int_\omega T(U_0).\nabla\eta,\\ &= \langle Q_{int}P, \eta\rangle + \langle Q_{trans}P, \eta\rangle + \langle Q_{ext}P, \eta\rangle, \quad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \end{aligned}$$

with

$$\begin{aligned} \langle Q_{int}P, \eta\rangle &:= -\gamma_1 \int_\omega \operatorname{div} T(U_0 + \nabla P)\eta,\\ \langle Q_{trans}P, \eta\rangle &:= \int_{\partial\omega} \left[\gamma_1 T(U_0 + (\nabla P)_{int}) - \gamma_0 T(U_0 + (\nabla P)_{ext})\right].n\ \eta,\\ \langle Q_{ext}P, \eta\rangle &:= -\gamma_0 \int_{\mathbb{R}^N\setminus\omega} \operatorname{div} T(U_0 + \nabla P)\eta. \end{aligned}$$

Hence P is a supersolution of Q in the half-space $\mathbb{R}^N_+$, that is

$$\langle QP, \eta\rangle \geq 0, \quad \forall \eta \in \mathcal{V}(\mathbb{R}^N), \operatorname{spt}(\eta) \subset \mathbb{R}^N_+, \eta \geq 0 \text{ a.e.}$$

if and only if the three following conditions are satisfied:

$$\begin{aligned} -\operatorname{div}(T(U_0 + \nabla P)) &\geq 0, && \forall x \in \omega \cap \mathbb{R}^N_+; && (4.18)\\ \left[\gamma_1 T(U_0 + (\nabla P)_{int}(x)) - \gamma_0 T(U_0 + (\nabla P)_{ext}(x))\right].n &\geq 0, && \forall x \in \partial\omega \cap \mathbb{R}^N_+; && (4.19)\\ -\operatorname{div}(T(U_0 + \nabla P)) &\geq 0, && \forall x \in \mathbb{R}^N_+ \setminus \bar{\omega}. && (4.20) \end{aligned}$$

As $W = W_a$, we denote for simplicity

$$T_a(\varphi) = \sigma(|\varphi|^2)\ \varphi, \quad \forall \varphi \in \mathbb{R}^N,$$

with

$$\sigma(\lambda) = \left(a^2 + \lambda\right)^{\frac{p-2}{2}}, \quad \forall \lambda \in \mathbb{R}_+.$$

In ω and in $\mathbb{R}^N \setminus \bar{\omega}$, it holds

$$\begin{aligned}\operatorname{div}(T_a(U_0+\nabla P)) &= \sigma(|U_0+\nabla P|^2)\ \Delta P + 2\sigma'(|U_0+\nabla P|^2)\ (U_0+\nabla P)^T\nabla^2 P(U_0+\nabla P)\\ &= \sigma(|U_0+\nabla P|^2)\left[\Delta P + \frac{p-2}{a^2+|U_0+\nabla P|^2}\ (U_0+\nabla P)^T\nabla^2 P(U_0+\nabla P)\right]. \quad (4.21)\end{aligned}$$

Thus, as $\sigma > 0$, the study of the sign in internal (resp. external) condition (4.18) (resp. (4.20)) can be carried out studying the sign of the term

$$\left(a^2+|U_0+\nabla P|^2\right)\Delta P + (p-2)(U_0+\nabla P)^T\nabla^2 P(U_0+\nabla P). \qquad (4.22)$$

(1) It is obvious that $\operatorname{div}(T_a(U_0+\nabla P)) = 0$ in ω. Thus internal condition (4.18) is satisfied.
(2) We now study external condition (4.20). For all $x \in \mathbb{R}^N, |x| > 1$ and all $\varphi \in \mathbb{R}^N$, denoting

$$r := |x| \quad \text{and } e_r := \frac{x}{|x|},$$

an easy calculation shows that

$$\nabla P(x) = kr^{-\beta}\left[U_0 - \beta\,(U_0.e_r)\,e_r\right] \qquad (4.23)$$

and

$$\varphi^T\nabla^2 P(x)\varphi \quad = \quad k\beta r^{-2-\beta}\left[(\beta+2)(U_0.x)\,(e_r.\varphi)^2 - 2(x.\varphi)(U_0.\varphi) - (U_0.x)\,|\varphi|^2\right]. \quad (4.24)$$

In particular one gets

$$\Delta P(x) \quad = \quad -k\beta r^{-2-\beta}(U_0.x)(N-\beta). \qquad (4.25)$$

As $\beta < N$, it follows that $\Delta P < 0$ in $\mathbb{R}^N_+\setminus\bar{\omega}$. In order to study the sign of (4.22), let us consider an arbitrary $x \in \mathbb{R}^N_+\setminus\bar{\omega}$. Obviously, if $(U_0+\nabla P(x))^T\nabla^2 P(x)(U_0+\nabla P(x)) \leq 0$, it follows immediately that

$$\left(a^2+|U_0+\nabla P(x)|^2\right)\Delta P(x) + (p-2)\ (U_0+\nabla P(x))^T\nabla^2 P(x)(U_0+\nabla P(x)) \leq 0,$$

therefore external condition (4.20) is satisfied at point x.

Hence, let us assume that $(U_0+\nabla P(x))^T\nabla^2 P(x)(U_0+\nabla P(x)) > 0$. Denoting

$$\tilde{\varphi} := U_0+\nabla P(x) \quad \text{and} \quad \cos\theta := \frac{x}{|x|}.\frac{U_0}{|U_0|} = e_r.\frac{U_0}{|U_0|},$$

one obtains

$$\begin{aligned} e_r.\tilde{\varphi} &= |U_0|\cos\theta\left[1+kr^{-\beta}(1-\beta)\right],\\ x.\tilde{\varphi} &= (U_0.x)\left[1+kr^{-\beta}(1-\beta)\right],\\ U_0.\tilde{\varphi} &= |U_0|^2\left[1+kr^{-\beta}(1-\beta\cos^2\theta)\right],\\ |\tilde{\varphi}|^2 &= |U_0|^2\left[\sin^2\theta\left(1+kr^{-\beta}\right)^2 + \cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right]. \end{aligned}$$

Thus formula (4.24) entails

$$\tilde{\varphi}^T\nabla^2 P(x)\tilde{\varphi} = k\beta r^{-2-\beta}(U_0.x)\,|U_0|^2\ f(r,\theta,k,\beta) \qquad (4.26)$$

with

$$\begin{aligned} f(r,\theta,k,\beta) &:= (\beta+1)\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\\ &\quad - 2\left(1+kr^{-\beta}(1-\beta)\right)\left(1+kr^{-\beta}(1-\beta\cos^2\theta)\right) - \sin^2\theta\left(1+kr^{-\beta}\right)^2. \end{aligned}$$

In addition formula (4.25) yields

$$\left(a^2+|\tilde{\varphi}|^2\right)\Delta P(x) = -k\beta r^{-2-\beta}(U_0.x)\,|U_0|^2\,(N-\beta) \times\left[\frac{a^2}{|U_0|^2}+\sin^2\theta\left(1+kr^{-\beta}\right)^2+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right]. \quad (4.27)$$

Hence the sign of (4.22) is negative if and only if the sign of

$$-(N-\beta)\left[\frac{a^2}{|U_0|^2}+\sin^2\theta\left(1+kr^{-\beta}\right)^2+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right]+(p-2)\;f(r,\theta,k,\beta) \quad (4.28)$$

is negative. As $p\geq 2$ and $\beta<N$, it thus suffices that

$$-(N-\beta)\left[\frac{a^2}{|U_0|^2}+\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\right]\;+$$
$$(p-2)\left[(\beta+1)\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2-2\left(1+kr^{-\beta}(1-\beta)\right)\left(1+kr^{-\beta}(1-\beta\cos^2\theta)\right)\right]$$

be negative. By inequality (4.16), it holds

$$1+kr^{-\beta}(1-\beta\cos^2\theta)\geq 1+kr^{-\beta}(1-\beta)\geq 1+k(1-\beta)>0.$$

It follows that

$$-2\left(1+kr^{-\beta}(1-\beta)\right)\left(1+kr^{-\beta}(1-\beta\cos^2\theta)\right)$$
$$\leq -2\left(1+kr^{-\beta}(1-\beta)\right)^2\leq -2\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2.$$

Hence it suffices that

$$-(N-\beta)\frac{a^2}{|U_0|^2}\;+\;\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2[\beta-N+(p-2)(\beta+1-2)]\;\leq\;0. \quad (4.29)$$

As $\beta>N/2\geq 1$, it follows from inequality (4.16) that

$$\cos^2\theta\left(1+kr^{-\beta}(1-\beta)\right)^2\leq 1,$$

Thus it suffices that

$$-(N-\beta)\frac{a^2}{|U_0|^2}+[\beta-N+(p-2)(\beta-1)]\leq 0,$$

which is equivalent to

$$\beta\leq\frac{N\left(1+\frac{a^2}{|U_0|^2}\right)+(p-2)}{1+\frac{a^2}{|U_0|^2}+(p-2)}. \quad (4.30)$$

There exists $\beta\in(N/2,N)$ satisfying inequality (4.30) as soon as

$$\frac{N}{2}<\frac{N\left(1+\frac{a^2}{|U_0|^2}\right)+(p-2)}{1+\frac{a^2}{|U_0|^2}+(p-2)}. \quad (4.31)$$

The latter condition (4.31) is equivalent to

$$p<2+\left(1+\frac{a^2}{|U_0|^2}\right)\frac{N}{N-2}=\bar{p},$$

with the convention that $\bar{p}=+\infty$ if $N=2$.

(3) Lastly let us prove that, for β chosen as above and k defined by (3.29), function P satisfies the transmission condition (4.20), that is

$$\gamma_1 T(U_0 + (\nabla P)_{int}(x)).x \geq \gamma_0 T(U_0 + (\nabla P)_{ext}(x)).x,\ \forall x, |x| = 1, U_0.\ x > 0.$$

As

$$T(\varphi) = \sigma(|\varphi|^2)\varphi,\ \forall \varphi \in \mathbb{R}^N$$

with σ an increasing function, a sufficient condition is given by the following three conditions:

$$\gamma_1 (U_0 + (\nabla P)_{int}(x)).x = \gamma_0 (U_0 + (\nabla P)_{ext}(x)).x, \qquad \forall x, |x| = 1, \tag{4.32}$$

$$|U_0 + (\nabla P)_{int}(x))| \geq |U_0 + (\nabla P)_{ext}(x))|\,, \qquad \forall x, |x| = 1, \tag{4.33}$$

$$(U_0 + (\nabla P)_{int}(x))\,.\ x \geq 0, \qquad \forall x, |x| = 1, U_0.x > 0. \tag{4.34}$$

(a) After the definition (3.28) of P, the first condition (4.32) reads

$$\gamma_1(1+k) = \gamma_0\,(1 + k(1-\beta))\,.$$

which exactly provides the value of k chosen in definition (3.29).

(b) After the definition (3.28) of P, the second condition (4.33) reads

$$(1+k)^2 \geq (1+k)^2 + k\beta\cos^2\theta\,(-2 + k(\beta - 2))\,,\ \forall \theta \in [-\pi/2, +\pi/2].$$

This condition is satisfied due to inequality (4.17).

(c) Regarding the latter condition (4.34), it holds

$$(U_0 + (\nabla P)_{int}(x)))\,.x = (1+k)U_0.x \geq 0.$$

The proof of Proposition 3.9 is now complete.

4.7. **Proof of proposition 3.11.** Recall that by symmetry there exists an element $\tilde{H}$ of the class H such that

$$\tilde{H}(x) = 0, \quad \forall x \in \mathbb{R}^N, U_0.\ x = 0.$$

Let P the supersolution defined in Proposition (3.9). For all $\eta \in \mathcal{V}(\mathbb{R}^N)$ such that $\mathrm{spt}(\eta) \subset \mathbb{R}^N_+$ and $\eta \geq 0$ a.e., we have obtained

$$\langle QP, \eta\rangle \geq 0.$$

As by definition of H, it holds $QH = 0$, it follows that

$$\langle QP - QH, \eta\rangle \geq 0,$$

that is

$$\int_{\mathbb{R}^N} \gamma\,[T(U_0 + \nabla P) - T(U_0 + \nabla H)]\,.\nabla\eta \geq 0. \tag{4.35}$$

As $P = \tilde{H} = 0$ in the hyperplane $(\mathbb{R}U_0)^\perp$, the test function defined by

$$\eta(x) := \begin{cases} \max(0, \tilde{H}(x) - P(x)) & \text{if } x \in \mathbb{R}^N_+, \\ 0 & \text{otherwise} \end{cases}$$

satisfies the conditions $\eta \in \mathcal{V}(\mathbb{R}^N)$, $\mathrm{spt}(\eta) \subset \mathbb{R}^N_+$ and $\eta \geq 0$. Hence it can be plugged into inequality (4.35). It follows

$$\int_{\{\tilde{H}>P\}\,\cap\,\mathbb{R}^N_+} \gamma\,[T(U_0 + \nabla P) - T(U_0 + \nabla H)]\,.(\nabla H - \nabla P) \geq 0.$$

Moreover after the ellipticity condition (5), there exists $c > 0$ such that

$$c\int_{\{\tilde{H}>P\}\,\cap\,\mathbb{R}^N_+} |\nabla H - \nabla P|^p + |\nabla H - \nabla P|^2$$
$$\leq \int_{\{\tilde{H}>P\}\,\cap\,\mathbb{R}^N_+} \gamma\,[T(U_0 + \nabla P) - T(U_0 + \nabla H)]\,.(\nabla P - \nabla H).$$

One concludes that

$$\int_{\mathbb{R}^N} |\nabla \eta|^p + |\nabla \eta|^2 = \int_{\{\tilde{H}>P\} \, \cap \, \mathbb{R}^N_+} |\nabla H - \nabla P|^p + |\nabla H - \nabla P|^2 = 0.$$

Due to the Poincaré inequality stated in Corollary (A.4), it follows that $\eta = 0$ in $\mathcal{V}(\mathbb{R}^N)$. Hence $P \geq \tilde{H}$ a.e. in $\mathbb{R}^N_+$.

Similarly, one easily obtains from Lemma 3.10 that $\tilde{H} \geq 0$ a.e. in $\mathbb{R}^N_+$.

We have thus obtained that $0 \leq \tilde{H} \leq P$ in $\mathbb{R}^N_+$. As $P \in L^\infty(\mathbb{R}^N)$ and

$$P(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty, y \in \mathbb{R}^N_+,$$

with $\tau := \beta - 1 > N/2 - 1$, one concludes that $\tilde{H} \in L^\infty(\mathbb{R}^N_+)$ with

$$\tilde{H}(y) = O\left(|y|^{-\tau}\right) \quad \text{as} \quad |y| \to +\infty, y \in \mathbb{R}^N_+.$$

Since $\tilde{H}$ is an odd function with respect to the first coordinate x_1, i.e. along the line $\mathbb{R}U_0$, the above properties immediately extend to the whole $\mathbb{R}^N$.

4.8. **Proof of Lemma 3.14.** As $H \in \mathcal{V}(\mathbb{R}^N)$, by definition it holds $\nabla H \in L^2(\mathbb{R}^N)$. Moreover according to Assumption 3.13, $\tilde{H} \in L^\infty(\mathbb{R}^N)$. As in addition $w_2 \in L^2(\mathbb{R}^N)$, it follows that $w_2\tilde{H} \in L^2(\mathbb{R}^N)$, which completes the proof of the assertion $H \in \mathcal{H}(\mathbb{R}^N)$.

4.9. **Proof of Proposition 3.15.** Let us begin proving a technical lemma. Recall $0 < \rho < R$ defined in (3.4) such that $\omega \subset\subset B(0,\rho) \subset B(0,R) \subset\subset \Omega$.

Let $\theta : \mathbb{R}^N \to \mathbb{R}$ a smooth function such that

$$\theta(x) = 0, \ \forall x \in \overline{B}(0,\rho) \quad \text{and} \quad \theta(x) = 1, \ \forall x \in \mathbb{R}^N \setminus B(0,R). \tag{4.36}$$

Recall

$$H_\varepsilon(x) := \varepsilon \tilde{H}(\varepsilon^{-1}x), \quad \forall x \in \Omega, \tag{4.37}$$

and set

$$\kappa_\varepsilon(x) = \theta(x) H_\varepsilon(x).$$

Lemma 4.2. *It holds $\kappa_\varepsilon \in W^{1,p}(\Omega)$ and $H_\varepsilon - \kappa_\varepsilon \in \mathcal{V}$. Moreover*

$$\|\nabla \kappa_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla \kappa_\varepsilon\|^2_{L^2(\Omega)} \;=\; o(\varepsilon^N). \tag{4.38}$$

Proof. Denote $C_\theta := \max(\|\theta\|_{L^\infty(\mathbb{R}^N)}, \|\nabla\theta\|_{L^\infty(\mathbb{R}^N)})$. Since $|\kappa_\varepsilon(x)| \leq C_\theta \, |H_\varepsilon(x)|$ for a.e. $x \in \Omega$, it follows from $H_\varepsilon \in L^p(\Omega)$ that $\kappa_\varepsilon \in L^p(\Omega)$. Next, from

$$\nabla \kappa_\varepsilon(x) = \nabla\theta(x) \ H_\varepsilon(x) + \theta(x) \ \nabla H_\varepsilon(x)$$

we infer by convexity

$$|\nabla \kappa_\varepsilon(x)|^p \leq 2^{p-1} C_\theta^p \left(|H_\varepsilon(x)|^p + |\nabla H_\varepsilon(x)|^p\right).$$

Thus $H_\varepsilon \in W^{1,p}(\Omega)$ entails that $\nabla\kappa_\varepsilon \in L^p(\Omega)$. One concludes $\kappa_\varepsilon \in W^{1,p}(\Omega)$. Moreover by definition of θ, it holds $H_\varepsilon - \kappa_\varepsilon = 0$ on $\partial\Omega$. Thus $H_\varepsilon - \kappa_\varepsilon \in W^{1,p}_0(\Omega) = \mathcal{V}$.

Let us now prove (4.38). Let $C := \max(2^{p-1}C_\theta^p, 2C_\theta^2)$. By convexity, for a.e. $x \in \Omega$ it holds

$$|\nabla \kappa_\varepsilon(x)|^p + |\nabla \kappa_\varepsilon(x)|^2 \leq C \left(|H_\varepsilon(x)|^p + |H_\varepsilon(x)|^2 + |\nabla H_\varepsilon(x)|^p + |\nabla H_\varepsilon(x)|^2\right). \tag{4.39}$$

(1) Since $\theta = 0$ in $B(0,\rho)$, we have

$$\int_{B(0,\rho)} |\nabla \kappa_\varepsilon|^p + |\nabla \kappa_\varepsilon|^2 = 0. \tag{4.40}$$

(2) Then let's integrate in $B(0,R)\setminus B(0,\rho)$. Making a change of scale and applying the asymptotic behavior of $\tilde{H}$ given by (3.33), we obtain

$$\begin{aligned}\int_{B(0,R)\setminus B(0,\rho)} |H_\varepsilon|^p + |H_\varepsilon|^2 &= \varepsilon^N \int_{B(0,R/\varepsilon)\setminus B(0,\rho/\varepsilon)} \varepsilon^p \left|\tilde{H}\right|^p + \varepsilon^2 \left|\tilde{H}\right|^2 \\ &\le \varepsilon^N\, O\left(\frac{R}{\varepsilon}\right)^N O\left(\varepsilon^p \left(\frac{\varepsilon}{\rho}\right)^{p\tau} + \varepsilon^2 \left(\frac{\varepsilon}{\rho}\right)^{2\tau}\right) \\ &\le O\left(\varepsilon^{p(1+\tau)} + \varepsilon^{2(1+\tau)}\right) = o(\varepsilon^N),\end{aligned}$$

since $p(1+\tau) \ge 2(1+\tau) > N$.

Recall $\nabla H \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus

$$\int_{B(0,R)\setminus B(0,\rho)} |\nabla H_\varepsilon|^p + |\nabla H_\varepsilon|^2 \le \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,\rho/\varepsilon)} |\nabla H|^p + |\nabla H|^2 = o\left(\varepsilon^N\right).$$

Therefore integrating inequality (4.39) in $B(0,R)\setminus B(0,\rho)$ entails

$$\int_{B(0,R)\setminus B(0,\rho)} |\nabla \kappa_\varepsilon|^p + |\nabla \kappa_\varepsilon|^2 = o(\varepsilon^N). \tag{4.41}$$

(3) Lastly it holds $\kappa_\varepsilon = H_\varepsilon$ in $\Omega \setminus \overline{B}(0,R)$ and thus $\nabla\kappa_\varepsilon = \nabla H_\varepsilon$ in $\Omega\setminus\overline{B}(0,R)$. Since $\nabla H \in L^p(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, it follows

$$\int_{\Omega\setminus B(0,R)} |\nabla\kappa_\varepsilon|^p + |\nabla\kappa_\varepsilon|^2 \le \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,R/\varepsilon)} |\nabla H|^p + |\nabla H|^2 = o\left(\varepsilon^N\right). \tag{4.42}$$

Gathering (4.40), (4.41) and (4.42), one eventually obtains (4.38). □

We now prove Proposition 3.15.

(1) First we prove inequality (3.36). For all $\eta \in \mathcal{V}$, define $\eta_\varepsilon \in \mathcal{V}(\mathbb{R}^N)$ by $\eta_\varepsilon(y) := \varepsilon^{-1}\eta(\varepsilon y)$ for all $y \in \Omega/\varepsilon$ and $\eta_\varepsilon(y) := 0$ for all $y \in \mathbb{R}^N\setminus(\Omega/\varepsilon)$. Applying variational formulation (3.19) to η_ε and making the change of scale backward, one obtains

$$\int_\Omega \gamma_\varepsilon \left[T(U_0+\nabla H_\varepsilon) - T(U_0)\right].\nabla\eta = -(\gamma_1-\gamma_0)\int_{\omega_\varepsilon} T(U_0).\nabla\eta. \tag{4.43}$$

Calculating the difference with variational form (3.18) yields

$$\int_\Omega \gamma_\varepsilon \left[T(U_0+\nabla h_\varepsilon) - T(U_0+\nabla H_\varepsilon)\right].\nabla\eta = 0, \quad \forall\, \eta\in\mathcal{V}. \tag{4.44}$$

Recall function κ_ε introduced in Lemma 4.2, such that $H_\varepsilon - \kappa_\varepsilon \in \mathcal{V}$. Plugging $\eta = h_\varepsilon - (H_\varepsilon - \kappa_\varepsilon) \in \mathcal{V}$ in (4.44) one obtains

$$\begin{aligned}&\int_\Omega \gamma_\varepsilon \left[T(U_0+\nabla h_\varepsilon) - T(U_0+\nabla H_\varepsilon)\right].(\nabla h_\varepsilon - \nabla H_\varepsilon) \\ &\qquad = -\int_\Omega \gamma_\varepsilon \left[T(U_0+\nabla h_\varepsilon) - T(U_0+\nabla H_\varepsilon)\right].\nabla\kappa_\varepsilon.\end{aligned} \tag{4.45}$$

Using condition (5), it follows that the left hand side of (4.45) can be bounded from below as

$$\begin{aligned}&\underline{\gamma} c\, \left(\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)}\right) \\ &\qquad \le \int_\Omega \gamma_\varepsilon \left[T(U_0+\nabla h_\varepsilon) - T(U_0+\nabla H_\varepsilon)\right].(\nabla h_\varepsilon - \nabla H_\varepsilon).\end{aligned} \tag{4.46}$$

Looking now at the right hand side of (4.45), applying inequality (3.1) with $M := |U_0| + \|\nabla H\|_{L^\infty(\mathbb{R}^N)}$, one obtains

$$\left|\int_\Omega [T(U_0+\nabla h_\varepsilon) - T(U_0+\nabla H_\varepsilon)] . \nabla\kappa_\varepsilon\right|$$
$$\le \int_\Omega \left[b_1 |\nabla h_\varepsilon - \nabla H_\varepsilon| + b_{p-1} |\nabla h_\varepsilon - \nabla H_\varepsilon|^{p-1}\right] . |\nabla\kappa_\varepsilon|$$
$$\le b_1 \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^2(\Omega)} \|\nabla\kappa_\varepsilon\|_{L^2(\Omega)} + b_{p-1} \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^{\frac{p}{q}}_{L^p(\Omega)} \|\nabla\kappa_\varepsilon\|_{L^p(\Omega)} . \quad (4.47)$$

Gathering (4.45), (4.46) and (4.47) as well as estimates (4.12) and (4.38), it follows that

$$\underline{\gamma} c \left(\|\nabla h_\varepsilon - \nabla H_\varepsilon\|^p_{L^p(\Omega)} + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|^2_{L^2(\Omega)}\right)$$
$$\le \overline{\gamma} b_1 \left(O(\varepsilon^N)\right)^{\frac{1}{2}} \left(o(\varepsilon^N)\right)^{\frac{1}{2}} + \overline{\gamma} b_{p-1} \left(O(\varepsilon^N)\right)^{\frac{1}{q}} \left(o(\varepsilon^N)\right)^{\frac{1}{p}} = o(\varepsilon^N),$$

which is the claimed estimate (3.36).

(2) We turn to the proof of inequality (3.37). Since $\nabla H \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and $r-1<0$ it holds

$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla H_\varepsilon|^p + |\nabla H_\varepsilon|^2 \quad \le \quad \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,\alpha\varepsilon^{r-1})} |\nabla H|^p + |\nabla H|^2 = o(\varepsilon^N).$$

The latter estimate combined with estimate (3.36) entails by convexity that

$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2 = o(\varepsilon^N)$$

which is the claimed estimate (3.37).

(3) We now prove estimate (3.38). After Lemma 3.6, ∇u_0 is β- Hölder continuous in a neighborhood of $x_0 = 0$ for some $\beta > 0$. Hence there exist $\delta > 0$ and $L > 0$ such that

$$|\nabla u_0(x) - U_0| \le L |x|^\beta, \quad \forall x \in B(0,\delta).$$

To apply estimate (3.37), we choose $\alpha := 1$ and $r = 1/2$. For all $\varepsilon \in (0, \delta^2)$, according to estimates (3.24) and (3.37) it follows

$$\int_\Omega |\nabla u_0 - U_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right)$$
$$\le \int_{B(0,\alpha\varepsilon^r)} L |x|^\beta \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) + 2 \|\nabla u_0\|_{L^\infty(\Omega)} \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right)$$
$$\le L\alpha^\beta \varepsilon^{r\beta}\, O(\varepsilon^N) + o(\varepsilon^N) = o(\varepsilon^N),$$

which completes the proof of estimate (3.38).

(4) For all $p \in (4,\infty)$ and for all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-2} \le \lambda^2 + \lambda^p$. Hence (3.39) follows immediately from (3.38).

(5) Similarly, for all $p \in (3,\infty)$ and for all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-1} \le \lambda^2 + \lambda^p$. Hence (3.40) follows immediately from (3.38).

(6) Regarding estimate (3.41), the Cauchy-Schwarz inequality and estimates (3.36), (3.24) and (3.25) result in

$$\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)$$
$$\le \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^2(\Omega)} \left[\|\nabla h_\varepsilon\|_{L^2(\Omega)} + \|\nabla H_\varepsilon\|_{L^2(\Omega)}\right] = o(\varepsilon^{\frac{N}{2}})\ O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N)$$

which completes the proof of estimate (3.41).

(7) Lastly let $p \in (3,\infty)$. For all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-2} \leq \lambda + \lambda^{p-1}$. Hence due to estimates (3.36), (3.24), (3.25), (3.41) and to Hölder's inequality one obtains

$$\begin{aligned}
&\int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-2} \\
&\qquad \leq \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right) + \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon| \left(|\nabla h_\varepsilon| + |\nabla H_\varepsilon|\right)^{p-1} \\
&\qquad\qquad \leq o(\varepsilon^N) + \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^p(\Omega)} \, \||\nabla h_\varepsilon| + |\nabla H_\varepsilon|\|_{L^p(\Omega)}^{\frac{p}{q}} \\
&\qquad\qquad\qquad = o(\varepsilon^N) + o(\varepsilon^{\frac{N}{p}}) \; O(\varepsilon^{\frac{N}{q}}) = o(\varepsilon^N)
\end{aligned}$$

which is the claimed estimate (3.42).

4.10. **Proof of Proposition 3.16.**

(1) Let us first prove inequality (3.43). For all $\eta \in \mathcal{V}$, calculating the difference between variational forms (3.17) and (3.18) yields after rearrangement

$$\begin{aligned}
\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta &= -(\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} (T(\nabla u_0) - T(U_0)).\nabla\eta \\
&+ \int_\Omega \gamma_\varepsilon \left[T(\nabla u_0) - T(U_0)\right].\nabla\eta + \int_\Omega \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta.
\end{aligned}$$

For all $\alpha > 0$ and for all $r \in (0,1)$, splitting the domains of integration of the two latter integrals into $B(0,\alpha\varepsilon^r)$ and $\Omega \setminus B(0,\alpha\varepsilon^r)$, one may rewrite the latter equality as

$$\begin{aligned}
\int_\Omega \gamma_\varepsilon \left[T(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta &= -(\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} (T(\nabla u_0) - T(U_0)).\nabla\eta \\
+ \int_{B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(\nabla u_0) - T(U_0)\right].\nabla\eta &+ \int_{B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta \\
+ \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(U_0)\right].\nabla\eta &+ \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(\nabla u_0) - T(\nabla u_0 + \nabla h_\varepsilon)\right].\nabla\eta.
\end{aligned}$$

Plugging the test function $\eta = \tilde{u}_\varepsilon - h_\varepsilon \in \mathcal{V}$ and applying condition (5) it follows that

$$c\underline{\gamma}\left(\|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^p(\Omega)}^p + \|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^2(\Omega)}^2\right) \quad \leq \quad \sum_{i=1}^5 \mathcal{E}_i(\varepsilon),$$

with

$$\mathcal{E}_1(\varepsilon) \quad = \quad -(\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} (T(\nabla u_0) - T(U_0)).(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.48}$$

$$\mathcal{E}_2(\varepsilon) \quad = \quad \int_{B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(\nabla u_0) - T(U_0)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.49}$$

$$\mathcal{E}_3(\varepsilon) \quad = \quad \int_{B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(\nabla u_0 + \nabla h_\varepsilon)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.50}$$

$$\mathcal{E}_4(\varepsilon) \quad = \quad \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(U_0 + \nabla h_\varepsilon) - T(U_0)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon), \tag{4.51}$$

$$\mathcal{E}_5(\varepsilon) \quad = \quad \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \gamma_\varepsilon \left[T(\nabla u_0) - T(\nabla u_0 + \nabla h_\varepsilon)\right].(\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon). \tag{4.52}$$

Hence it suffices to prove that there exist $\alpha > 0$ and $r \in (0,1)$ such that

$$\mathcal{E}_i(\varepsilon) = o(\varepsilon^N), \quad \forall i, 1 \leq i \leq 5.$$

After Lemma 3.6 there exists $\beta > 0$ such that ∇u_0 is β-Hölder continuous in a neighborhood of $x_0 = 0$. To apply estimate (3.37), we choose $\alpha := \rho$ (see (3.4)) and

$$r := \frac{1}{2}\left(\frac{N}{2\beta + N} + 1\right) \in (0, 1).$$

In particular after (3.4), it holds $\omega \subset\subset B(0, \rho) = B(0, \alpha)$. In addition, after condition (1), T is Lipschitz-continuous in a neighborhood of U_0. Hence there exist $\delta > 0$ and $L > 0$ such that

$$\max(|\nabla u_0(x) - U_0|\,, |T(\nabla u_0(x)) - T(U_0)|) \leq L\,|x|^\beta \quad \forall x \in \Omega, |x| \leq \delta. \tag{4.53}$$

In addition it holds

$$\omega_\varepsilon \subset B(0, \rho\varepsilon) \subset B(0, \rho\varepsilon^r) \subset B(0, \delta) \qquad 0 < \varepsilon < \min\left(1, \left(\frac{\delta}{\rho}\right)^{\frac{1}{r}}\right).$$

(a) Applying Cauchy-Schwarz's inequality, it follows from estimates (4.53) and (4.11) that

$$\begin{aligned} |\mathcal{E}_1(\varepsilon)| &\leq 2\overline{\gamma}\int_{\omega_\varepsilon} |(T(\nabla u_0) - T(U_0))|\,|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\leq 2\overline{\gamma}L \times O\left(\varepsilon^{\beta + \frac{N}{2}}\right) \times O\left(\varepsilon^{\frac{N}{2}}\right) = o(\varepsilon^N). \end{aligned}$$

(b) Similarly after estimate (4.11) and (4.53) and Cauchy-Schwarz's inequality, it holds

$$|\mathcal{E}_2(\varepsilon)| \leq \overline{\gamma}L \times O\left(\varepsilon^{r(\beta + \frac{N}{2})}\right) \times O\left(\varepsilon^{\frac{N}{2}}\right) = O\left(\varepsilon^{N + \frac{\beta}{2}}\right) = o(\varepsilon^N).$$

(c) After condition (6), it holds

$$|\mathcal{E}_3(\varepsilon)| \leq \overline{\gamma}C\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\left[1 + |\nabla u_0 + \nabla h_\varepsilon|^{p-2} + |U_0 - \nabla u_0|^{p-2}\right]|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon|\,. \tag{4.54}$$

Let us look for an upper bound for (4.54).

- First after estimates (4.11) and (4.53) and Cauchy-Schwarz's inequality it holds

$$\begin{aligned} &\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\left[1 + |U_0 - \nabla u_0|^{p-2}\right]|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\qquad \leq L\rho^\beta\varepsilon^{\beta r}\left[1 + \left(L\delta^\beta\right)^{p-2}\right]\int_{B(0,\rho\varepsilon^r)} |\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\qquad \leq \varepsilon^{\beta r} O\left(\varepsilon^{\frac{rN}{2}}\right)\; O\left(\varepsilon^{\frac{N}{2}}\right) = O\left(\varepsilon^{\frac{r(2\beta+N)+N}{2}}\right) = O\left(\varepsilon^{N+\frac{\beta}{2}}\right) = o(\varepsilon^N). \end{aligned}$$

- Then, since $p \geq 2$,

$$\begin{aligned} &\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\,|\nabla u_0 + \nabla h_\varepsilon|^{p-2}\,|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon| \\ &\qquad \leq 2^{p-2}\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|\,(|\nabla u_0|^{p-2} + |\nabla h_\varepsilon|^{p-2})\,|\nabla\tilde{u}_\varepsilon - \nabla h_\varepsilon|\,. \end{aligned}$$

On one hand, after estimates (4.11) and (4.53) and Cauchy-Schwarz's inequality, it holds

$$\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0| \, |\nabla u_0|^{p-2} \, |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon|$$
$$\leq L\rho^\beta \varepsilon^{\beta r} \left(|U_0| + L\delta^\beta\right)^{p-2} O\left(\varepsilon^{\frac{rN}{2}}\right) \, O\left(\varepsilon^{\frac{N}{2}}\right) = O\left(\varepsilon^{\frac{r(2\beta+N)+N}{2}}\right) = o(\varepsilon^N).$$

On the other hand, according to (4.53), (3.24) and (4.11) and Hölder's inequality

$$\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0| \, |\nabla h_\varepsilon|^{p-2} \, |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon|$$
$$\leq \left(\int_{B(0,\rho\varepsilon^r)} |U_0 - \nabla u_0|^p\right)^{\frac{1}{p}} \left(\int_{B(0,\rho\varepsilon^r)} |\nabla h_\varepsilon|^p\right)^{\frac{p-2}{p}} \left(\int_{B(0,\rho\varepsilon^r)} |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon|^p\right)^{\frac{1}{p}}$$
$$\leq O\left(\varepsilon^{r(\beta p+N)}\right)^{\frac{1}{p}} O(\varepsilon^N)^{\frac{p-2}{p}} O(\varepsilon^N)^{\frac{1}{p}} = O\left(\varepsilon^s\right) = o(\varepsilon^N).$$

since

$$s = N + \frac{r(\beta p + N) - N}{p} = N + \frac{1}{p}\left[\frac{1}{2}\left(\frac{N}{2\beta+N} + 1\right)(\beta p + N) - N\right]$$
$$\geq N + \frac{1}{p}\left[\frac{1}{2}\left(\frac{N}{2\beta+N} + 1\right)(2\beta + N) - N\right] = N + \frac{\beta}{p}.$$

Hence the upper bound of (4.54) is a $o(\varepsilon^N)$, and one concludes from (4.54) that

$$\mathcal{E}_3(\varepsilon) \quad = \quad o(\varepsilon^N).$$

(d) After inequality (3.1) it holds

$$|\mathcal{E}_4(\varepsilon)| \leq \overline{\gamma} \int_{\Omega\setminus B(0,\rho\varepsilon^r)} \left[b_1 \, |\nabla h_\varepsilon| + b_{p-1} \, |\nabla h_\varepsilon|^{p-1}\right] |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \, .$$

Applying Hölder's inequality and estimates (4.11) and (3.37), it follows

$$\begin{aligned} |\mathcal{E}_4(\varepsilon)| \quad &\leq \quad \overline{\gamma} b_1 \left(\int_{\Omega\setminus B(0,\rho\varepsilon^r)} |\nabla h_\varepsilon|^2\right)^{\frac{1}{2}} \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^2(\Omega)} \\ &\quad + \overline{\gamma} b_{p-1} \left(\int_{\Omega\setminus B(0,\rho\varepsilon^r)} |\nabla h_\varepsilon|^p\right)^{\frac{1}{q}} \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^p(\Omega)} \\ &\leq \quad o(\varepsilon^{\frac{N}{2}}) \, O(\varepsilon^{\frac{N}{2}}) + o(\varepsilon^{\frac{N}{q}}) \, O(\varepsilon^{\frac{N}{p}}) = o(\varepsilon^N). \end{aligned}$$

(e) After Lemma 3.6 it holds $\nabla u_0 \in L^\infty(\Omega)$. Thus one can apply again inequality (3.1) and proves exactly as for $\mathcal{E}_4(\varepsilon)$ that $\mathcal{E}_5(\varepsilon) = o(\varepsilon^N)$.

(2) Regarding estimate (3.44), Cauchy-Schwarz's inequality and estimates (3.43), (3.23) and (3.24) entail that

$$\int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)$$
$$\leq \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^2(\Omega)} \left(\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)} + \|\nabla h_\varepsilon\|_{L^2(\Omega)}\right) = o(\varepsilon^{\frac{N}{2}}) \, O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N),$$

which proves estimate (3.44).

(3) Let $p \in (3,\infty)$. For all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-2} \le \lambda + \lambda^{p-1}$. Hence due to estimates (3.44), (3.43), (3.23), (3.24) and to Hölder's inequality one obtains

$$\begin{aligned}
&\int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-2} \\
&\qquad \le \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right) + \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| \left(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\right)^{p-1} \\
&\le o(\varepsilon^N) + \|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^p(\Omega)} \, \||\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|\|^{\frac{p}{q}}_{L^p(\Omega)} \,, = o(\varepsilon^N) + o(\varepsilon^{\frac{N}{p}}) \; O(\varepsilon^{\frac{N}{q}}) = o(\varepsilon^N)
\end{aligned}$$

which is the claimed estimate (3.45).

(4) Eventually estimate (3.46) immediately follows by convexity from estimates (3.37) and (3.43).

4.11. **Proof of Lemma 3.17.** As $\mathcal{J}_\varepsilon$ is assumed to be the compliance, it holds $G = f \in C^{0,\alpha}(\Omega)$. The unperturbed adjoint state $v_0 \in \mathcal{H}$ is weak solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\gamma_0 DT(\nabla u_0)\nabla v_0\right) = -f & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.55}$$

According to condition (1) it holds $DT \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R})$, and by Lemma 3.6 it holds $\nabla u_0 \in C^{0,\beta}(\overline{\Omega})$. Hence $DT(\nabla u_0) \in C^{0,\alpha\beta}(\overline{\Omega})$. In addition, $DT(\nabla u_0)$ is uniformly strictly 2-elliptic according to the lower bound of condition (4), and we have $f \in C^{0,\alpha}(\overline{\Omega})$. Therefore according to [34] Thm 8.34, problem (4.55) admits a unique strong solution $w_0 \in C^{1,\tilde{\beta}}(\overline{\Omega})$ with $\tilde{\beta} = \min(\alpha, \alpha\beta) > 0$. According to the weak maximum principle, [34] Cor 8.2, one has $v_0 = w_0$. As Ω is bounded and as by definition $v_0 \in \mathcal{H}$, it follows that $v_0 \in L^\infty(\Omega)$, $\nabla v_0 \in L^\infty(\Omega)$ and $v_0 \in \mathcal{V}$.

4.12. **Proof of Lemma 3.19.**

(1) We first prove estimate (3.52). After Lemma 3.6, it holds $u_0 \in C^{1,\beta}(\overline{\Omega})$, hence $\nabla u_0 \in L^\infty(\overline{\Omega})$ and $DT(\nabla u_0) \in L^\infty(\Omega)$. Moreover we assumed $\nabla v_0 \in L^\infty(\Omega)$. Due to the ellipticity of DT stated in condition (4), applying test function $\eta = \tilde{v}_\varepsilon$ in the variational form (3.48) defining $\tilde{v}_\varepsilon$, it holds:

$$\begin{aligned}
\underline{\gamma}\; c \int_\Omega |\nabla \tilde{v}_\varepsilon|^2 \;&\le\; \int_\Omega \gamma_\varepsilon DT(\nabla u_0)\nabla \tilde{v}_\varepsilon . \nabla \tilde{v}_\varepsilon = -\left(\gamma_1 - \gamma_0\right) \int_{\omega_\varepsilon} DT(\nabla u_0)\nabla v_0 . \nabla \tilde{v}_\varepsilon \\
&\le\; |\gamma_1 - \gamma_0| \, \|DT(\nabla u_0)\|_{L^\infty} \|\nabla v_0\|_{L^\infty} \int_{\omega_\varepsilon} |\nabla \tilde{v}_\varepsilon| \\
&\le\; |\gamma_1 - \gamma_0| \, \|DT(\nabla u_0)\|_{L^\infty} \|\nabla v_0\|_{L^\infty} \, |\omega|^{\frac{1}{2}} \, \varepsilon^{\frac{N}{2}} \;\; \|\nabla \tilde{v}_\varepsilon\|_{L^2(\Omega)} \,.
\end{aligned}$$

This provides estimate (3.52).

(2) The upper bound (3.53) to $\|\nabla k_\varepsilon\|^2_{L^2(\Omega)}$ is obtained in the same way.

(3) After a change of scale and since by definition $\nabla K \in L^2(\mathbb{R}^N)$ it holds

$$\|\nabla K_\varepsilon\|^2_{L^2(\Omega)} = \varepsilon^N \int_{\Omega/\varepsilon} |\nabla K|^2 \le \varepsilon^N \int_{\mathbb{R}^N} |\nabla K|^2 = O(\varepsilon^N),$$

which is estimate (3.54).

4.13. **Proof of Proposition 3.20.**

(1) We first study the asymptotic behavior of K and ∇K. The variational form (3.50) can be rewritten

$$\int_{\mathbb{R}^N} DT(U_0)\nabla K . \nabla \eta = \left(1 - \frac{\gamma_1}{\gamma_0}\right) \int_\omega DT(U_0)(V_0 + \nabla K).\nabla \eta, \quad \forall\, \eta \in \mathcal{H}(\mathbb{R}^N). \tag{4.56}$$

As matrix $DT(U_0)$ is positive definite, up to a linear change of coordinates in $\mathbb{R}^N$, one can asume that $DT(U_0) = I_N$.

The proof of the asymptotic behavior of K is standard (e.g. [8], §3.1.2). We denote by E the fundamental solution of the Laplace operator in $\mathbb{R}^N$, given for all $y \in \mathbb{R}^N$, $y \neq 0$ by

$$E(y) := \begin{cases} \frac{1}{2\pi} \ \log|y|, & \text{if } N = 2, \\ \frac{1}{(2-N)A_N} \ |y|^{2-N}, & \text{if } N \geq 3, \end{cases} \tag{4.57}$$

with A_N the area of the unit sphere of $\mathbb{R}^N$. In particular it holds

$$\forall N \geq 2, \exists C_N > 0, \forall y \in \mathbb{R}^N, y \neq 0, \quad |\nabla E(y)| \leq C_N |y|^{1-N}. \tag{4.58}$$

Denote by $\mathcal{T}$ the distribution in $\mathbb{R}^N$ defined by

$$\langle \mathcal{T}, \eta \rangle := \left(\frac{\gamma_1}{\gamma_0} - 1 \right) \int_\omega (V_0 + \nabla K).\nabla \eta, \quad \forall \eta \in C_0^\infty(\mathbb{R}^N).$$

It follows from (4.56) that

$$\Delta K = \mathcal{T}.$$

Hence consider the element $\tilde{K}$ of the class K given by

$$\tilde{K} = \mathcal{T} * E. \tag{4.59}$$

Let $\rho > 0$ such that $\omega \subset B(0, \rho)$. To study the behavior of $\tilde{K}$ at infinity, let $y \in \mathbb{R}^N$, $|y| \geq 2\rho$. The convolution (4.59) reads

$$\tilde{K}(y) = \left(\frac{\gamma_1}{\gamma_0} - 1 \right) \int_\omega (V_0 + \nabla K(z)).\nabla E(y - z) \, dz. \tag{4.60}$$

Since $V_0 + \nabla K \in L^2(\omega)$, the Cauchy-Schwarz's inequality yields

$$\left| \tilde{K}(y) \right| \leq C \left(\int_\omega |\nabla E(y - z)|^2 \ dz \right)^{\frac{1}{2}},$$

with $C := \left| \frac{\gamma_1}{\gamma_0} - 1 \right| \ \|V_0 + \nabla K\|_{L^2(\omega)}$. In addition, (4.58) yields

$$\begin{aligned} \int_\omega |\nabla E(y - z)|^2 \ dz &\leq C_N^2 \int_\omega |y - z|^{2-2N} \leq C_N^2 \int_\omega (|y| - |z|)^{2-2N} \\ &\leq C_N^2 \, |\omega| \left(\frac{1}{2} \right)^{2-2N} |y|^{2-2N}. \end{aligned}$$

Hence

$$\left| \tilde{K}(y) \right| \leq C' \, |y|^{1-N},$$

with $C' = C \ C_N \ 2^{N-1} \, |\omega|^{\frac{1}{2}}$. This proves (3.55).

The calculations proving the asymptotic behavior (3.56) of ∇K are similar, based on $\nabla K = \mathcal{T} * \nabla E$.

(2) As by definition it holds $\nabla K \in L^2(\mathbb{R}^N)$, the claimed regularity $K \in \mathcal{V}(\mathbb{R}^N)$ is equivalent to $w_p \tilde{K} \in L^p(\mathbb{R}^N)$ and $\nabla K \in L^p(\mathbb{R}^N)$. If $\omega \subset\subset B(0, M)$, the fact that $w_p \tilde{K} \in L^p(\mathbb{R}^N \setminus B(0, M))$ and $\nabla K \in L^p(\mathbb{R}^N \setminus B(0, M))$ is a straightforward consequence of the estimates (3.55) and (3.56). The variational form (3.50) defining K can be rewritten in the strong form

$$\begin{cases} -\operatorname{div}(DT(U_0)K) = 0 & \text{in } \mathbb{R}^N \setminus \partial\omega, \\ \gamma_0 \left(DT(U_0) \nabla K.n \right)_{ext} - \gamma_1 \left(DT(U_0) \nabla K.n \right)_{int} = (\gamma_1 - \gamma_0)(DT(U_0)V_0).n & \text{on } \partial\omega. \end{cases} \tag{4.61}$$

Such transmission problems, with a source of zero mean value on $\partial\omega$, have been studied e.g. [6], §2.4. The solution is a single layer potential. As $\partial\omega$ is C^2 and the source is continuous, the regularity of the density entails that $K \in L^\infty(B(0,M))$ and $\nabla K \in L^\infty(B(0,M))$. Hence $w_p\tilde{K} \in L^p(B(0,M))$ and $\nabla K \in L^p(B(0,M))$. One concludes that $w_p\tilde{K} \in L^p(\mathbb{R}^N)$ and $\nabla K \in L^p(\mathbb{R}^N)$, i.e., $K \in \mathcal{V}(\mathbb{R}^N)$.

4.14. **Proof of Lemma 3.21.** We start proving a technical lemma. Consider a smooth function $\theta : \mathbb{R}^N \to \mathbb{R}$ such that

$$\theta(x) = 0, \ \forall x \in B(0,\rho) \quad \text{and} \quad \theta(x) = 1, \ \forall x \in \mathbb{R}^N \setminus B(0,R)$$

where $0 < \rho < R$ were defined in (3.4), that is $\omega \subset\subset B(0,\rho) \subset B(0,R) \subset\subset \Omega \setminus \mathrm{spt}(f)$. Denote

$$C_\theta := \sup\left\{\max(|\theta(x)|\,, |\nabla\theta(x)|\,; x \in \mathbb{R}^N\right\} < \infty.$$

Recall function K_ε is defined by (3.51), and set

$$K_{\theta,\varepsilon}(x) = \theta(x)K_\varepsilon(x).$$

According to the Leibniz formula, for a.e. $x \in \Omega$ it holds

$$|\nabla K_{\theta,\varepsilon}(x)|^2 \le 2C_\theta^2\left(|K_\varepsilon(x)|^2 + |\nabla K_\varepsilon(x)|^2\right).$$

Since $K_\varepsilon \in H^1(\Omega)$, it follows $K_{\theta,\varepsilon} \in H^1(\Omega)$.

Lemma 4.3. *It holds*

$$\|\nabla K_{\theta,\varepsilon}\|^2_{L^2(\Omega)} \quad = \quad o(\varepsilon^N). \tag{4.62}$$

Proof. As $\theta = 0$ in $B(0,\rho)$, we have of course

$$\int_{B(0,\rho)} |\nabla K_{\theta,\varepsilon}|^2 = 0. \tag{4.63}$$

Integrating in $B(0,R) \setminus B(0,\rho)$, according to the asymptotic behavior of K given by (3.55) and since $\nabla K \in L^2(\mathbb{R}^N)$ one obtains

$$\begin{aligned}\frac{1}{2C_\theta^2}\int_{B(0,R)\setminus B(0,\rho)} |\nabla K_{\theta,\varepsilon}|^2 &\le \int_{B(0,R)\setminus B(0,\rho)} |K_\varepsilon|^2 \ + \int_{B(0,R)\setminus B(0,\rho)} |\nabla K_\varepsilon|^2 \\ &\le \varepsilon^{2+N}\int_{B(0,R/\varepsilon)\setminus B(0,\rho/\varepsilon)} \left|\tilde{K}\right|^2 \ + \ \varepsilon^N \int_{B(0,R/\varepsilon)\setminus B(0,\rho/\varepsilon)} |\nabla K|^2 \\ &\le \varepsilon^{N+2}\ O\left(\left(\frac{\rho}{\varepsilon}\right)^{2-2N}\left(\frac{R}{\varepsilon}\right)^N\right) \ + \varepsilon^N o(1) = o(\varepsilon^N).\end{aligned} \tag{4.64}$$

Lastly it holds $K_{\theta,\varepsilon} = K_\varepsilon$ in $\Omega \setminus B(0,R)$. Again $\nabla K \in L^2(\mathbb{R}^N)$ and thus

$$\int_{\Omega\setminus B(0,R)} |\nabla K_{\theta,\varepsilon}|^2 \le \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,R/\varepsilon)} |\nabla K|^2 = o\left(\varepsilon^N\right). \tag{4.65}$$

Gathering (4.63), (4.64) and (4.65), one obtains the claimed estimate (4.62). □

We now prove Lemma 3.21.

(1) We begin proving estimate (3.58). For all $\eta \in \mathcal{H}$, we define $\eta_1 \in \mathcal{H}(\mathbb{R}^N)$ by

$$\eta_1(y) := \frac{1}{\varepsilon}\eta(\varepsilon y), \ \forall y \in \Omega/\varepsilon \quad \text{and} \quad \eta_1(y) := 0, \ \forall y \in \mathbb{R}^N \setminus (\Omega/\varepsilon).$$

Applying the variational form (3.50) to $\eta_1 \in \mathcal{H}(\mathbb{R}^N)$ and making the change of scale backward, one obtains

$$\int_\Omega \gamma_\varepsilon DT(U_0)\nabla K_\varepsilon.\nabla\eta = -\left(\gamma_1 - \gamma_0\right)\int_{\omega_\varepsilon} DT(U_0)V_0.\nabla\eta, \ \forall \eta \in \mathcal{H}.$$

Then calculating the difference with the variational form (3.49) yields

$$\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).\nabla\eta = 0,\ \forall \eta \in \mathcal{H}. \tag{4.66}$$

Using that $K_{\theta,\varepsilon} \in H^1(\Omega)$ and $K_\varepsilon - K_{\theta,\varepsilon} \in \mathcal{H}$, choosing $\eta = k_\varepsilon - (K_\varepsilon - K_{\theta,\varepsilon}) \in \mathcal{H}$ in (4.66), we arrive at

$$\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).(\nabla k_\varepsilon - \nabla K_\varepsilon) = -\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).\nabla K_{\theta,\varepsilon}.$$

Then applying condition (4) one obtains

$$\begin{aligned}\underline{\gamma}\ c\int_\Omega |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 \ &\leq\ \left|\int_\Omega \gamma_\varepsilon DT(U_0)(\nabla k_\varepsilon - \nabla K_\varepsilon).\nabla K_{\theta\varepsilon}\right| \\ &\leq\ \overline{\gamma}\,|DT(U_0)| \left(\int_\Omega |\nabla k_\varepsilon - \nabla K_\varepsilon|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla K_{\theta,\varepsilon}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

This implies

$$\|\nabla k_\varepsilon - \nabla K_\varepsilon\|^2_{L^2(\Omega)} \ \leq\ \left(\overline{\gamma}\,|DT(U_0)|\,/\underline{\gamma}\ c\right)^2 \|\nabla K_{\theta,\varepsilon}\|^2_{L^2(\Omega)}\,.$$

Using (4.62) completes the proof of (3.58).

(2) Let us now prove (3.59). Let $\alpha > 0$ and $r \in (0,1)$. By convexity we have

$$|\nabla k_\varepsilon|^2 \leq 2\,|\nabla k_\varepsilon - \nabla K_\varepsilon|^2 + 2\,|\nabla K_\varepsilon|^2\,.$$

After a change of scale one obtains

$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon|^2 \ \leq\ 2\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon - \nabla K_\varepsilon|^2\ +\ 2\ \ \varepsilon^N \int_{\mathbb{R}^N\setminus B(0,\alpha\varepsilon^{r-1})} |\nabla K|^2. \tag{4.67}$$

After (3.58) it holds

$$\int_{\Omega\setminus B(0,\alpha\varepsilon^r)} |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 \leq \int_\Omega |\nabla k_\varepsilon - \nabla K_\varepsilon|^2 = o(\varepsilon^N).$$

Again $\nabla K \in L^2(\mathbb{R}^N)$ and $r-1<0$ entail that

$$\int_{\mathbb{R}^N\setminus B(0,\alpha\varepsilon^{r-1})} |\nabla K|^2 = o(1).$$

Hence (4.67) yields (3.59).

4.15. **Proof of Lemma 3.22.** Recall that, after Lemma 3.6, ∇u_0 is β-Hölder continuous in $\bar{\Omega}$ for some $\beta > 0$. Also, after condition (1), DT is $\tilde{\alpha}$-Hölder continuous on every compact set for some $\tilde{\alpha} > 0$, and by assumption ∇v_0 is $\tilde{\beta}$-Hölder continuous in a neighborhood of point $x_0 = 0$ for some $\tilde{\beta} > 0$. Let $\tilde{\tau} := \min(\tilde{\alpha}\beta, \tilde{\beta}) > 0$. Hence there exist $\delta \in (0,1)$ and $L > 0$ such that for all $x \in B(0,\delta)$ it holds

$$|DT(\nabla u_0(x)) - DT(U_0)| + |DT(\nabla u_0(x))\nabla v_0(x) - DT(U_0)V_0| \leq L\,|x|^{\tilde{\tau}}\,. \tag{4.68}$$

Let $\rho > 0$ be such that $\omega \subset B(0,\rho)$, see (3.4). So as to apply estimate (3.59), we choose $\alpha := \rho$ and $r := 1/2$. Lastly, for all $\varepsilon < \min(1, (\delta/\rho)^2)$ it holds

$$\omega_\varepsilon \subset B(0,\rho\varepsilon) \subset B(0,\rho\varepsilon^r) \subset B(0,\delta).$$

We can now start our estimations. According to condition (4), it holds

$$\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|^2_{L^2(\Omega)} \leq \frac{1}{\underline{\gamma} c}\int_\Omega \gamma_\varepsilon DT(\nabla u_0)\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)^2\,. \tag{4.69}$$

Calculating the difference between the variational forms (3.48) and (3.49) and choosing $\eta = \tilde{v}_\varepsilon - k_\varepsilon \in \mathcal{H}$, one obtains:

$$\begin{aligned}\int_\Omega &\gamma_\varepsilon DT(\nabla u_0)\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)^2 \\ &= -\,(\gamma_1-\gamma_0)\int_{\omega_\varepsilon}(DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0)\,.\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon) \\ &\qquad - \int_\Omega (\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0))\,\nabla k_\varepsilon.\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)\,. \qquad (4.70)\end{aligned}$$

(1) Regarding the first term on the right-hand side of (4.70), it follows from (4.68) and from Cauchy-Schwarz's inequality that

$$\begin{aligned}&\left|\int_{\omega_\varepsilon}(DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0)\,.\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)\right| \\ &\qquad \le L\rho^{\tilde{\tau}}\varepsilon^{\tilde{\tau}}\,|\omega|^{\frac12}\,\varepsilon^{\frac N2}\,\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} = C_1\varepsilon^{\tilde{\tau}+\frac N2}\,\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} \qquad (4.71)\end{aligned}$$

where C_1 is a positive constant.

(2) Then consider the second term on the right side of (4.70). We split the domain of integration into $B(0,\alpha\varepsilon^r)$ and $\Omega\setminus B(0,\alpha\varepsilon^r)$. Applying (4.68), the Cauchy-Schwarz inequality and estimate (3.53), we arrive at

$$\begin{aligned}&\left|\int_{B(0,\alpha\varepsilon^r)}(\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0))\,\nabla k_\varepsilon.\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)\right| \\ &\qquad \le \overline{\gamma}L\alpha^{\tilde{\tau}}\varepsilon^{r\tilde{\tau}}C^{\frac12}\varepsilon^{\frac N2}\,\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} = C_2\varepsilon^{r\tilde{\tau}+\frac N2}\,\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} \qquad (4.72)\end{aligned}$$

with C_2 a positive constant. Regarding the integral in $\Omega\setminus B(0,\alpha\varepsilon^r)$, the term $\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0)$ is bounded by $\tilde{C} := 2\overline{\gamma}\,\|DT(\nabla u_0)\|_{L^\infty(\Omega)}$. After Cauchy-Schwarz's inequality and estimate (3.59) one obtains

$$\begin{aligned}&\left|\int_{\Omega\setminus B(0,\alpha\varepsilon^r)}(\gamma_\varepsilon DT(\nabla u_0) - \gamma_\varepsilon DT(U_0))\,\nabla k_\varepsilon\,(\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon)\right| \\ &\qquad \le \tilde{C}\left(\int_{\Omega\setminus B(0,\alpha\varepsilon^r)}|\nabla k_\varepsilon|^2\right)^{\frac12}.\,\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} = o(\varepsilon^{\frac N2})\;\;\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)}\,. \qquad (4.73)\end{aligned}$$

Therefore, gathering (4.69), (4.70), (4.71), (4.72) and (4.73) and dividing by $\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)}$, it follows that

$$\underline{\gamma}c\,\|\nabla\tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} \;=\; o(\varepsilon^{\frac N2}),$$

and (3.60) is proven.

4.16. **Proof of Lemma 3.23.** It follows from definitions (3.68) and (3.71) that

$$\begin{aligned}\tilde{j}_1(\varepsilon) - \varepsilon^N J_1 &= (\gamma_1-\gamma_0)\left[\int_{\omega_\varepsilon}T(U_0).(V_0+\nabla k_\varepsilon) - \varepsilon^N\int_\omega T(U_0).(V_0+\nabla K)\right] \\ &= (\gamma_1-\gamma_0)\;T(U_0).\int_{\omega_\varepsilon}(\nabla k_\varepsilon - \nabla K_\varepsilon).\end{aligned}$$

Using (3.58) and Cauchy-Schwarz's inequality, we obtain

$$\left|\tilde{j}_1(\varepsilon) - \varepsilon^N J_1\right| \;\le\; 2\overline{\gamma}\,|T(U_0)|\,|\omega|^{\frac12}\,\varepsilon^{\frac N2}\,\|\nabla k_\varepsilon - \nabla K_\varepsilon\|_{L^2(\Omega)} \le O(\varepsilon^{\frac N2})\;o(\varepsilon^{\frac N2}) = o(\varepsilon^N),$$

which proves Lemma 3.23.

4.17. **Proof of Lemma 3.24.** It follows from definitions (3.66) and (3.68) that

$$\begin{aligned}
j_1(\varepsilon) - \tilde{j}_1(\varepsilon) &= (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} [T(\nabla u_0).\nabla v_\varepsilon - T(U_0).(V_0 + \nabla k_\varepsilon)] \\
&= (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_0 - T(U_0).V_0 \\
&\quad + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(\nabla u_0).\nabla \tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon.
\end{aligned}$$

Since $x \in \Omega \mapsto T(\nabla u_0(x)).\nabla v_0(x)$ is continuous at point $x_0 = 0$, it holds

$$\int_{\omega_\varepsilon} T(\nabla u_0).\nabla v_0 - T(U_0).V_0 \;=\; |\omega_\varepsilon|\, o(1) = o(\varepsilon^N).$$

Moreover, since $x \in \Omega \mapsto T(\nabla u_0(x))$ is continuous at point $x_0 = 0$, after Cauchy-Schwarz's inequality and estimates (3.60) and (3.53), it holds

$$\begin{aligned}
&\int_{\omega_\varepsilon} |T(\nabla u_0).\nabla \tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon| \\
&\qquad \leq \int_{\omega_\varepsilon} |T(\nabla u_0)|\, |\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon| + \int_{\omega_\varepsilon} |T(\nabla u_0) - T(U_0)|\, |\nabla k_\varepsilon| \\
&\qquad\qquad \leq |\omega|^{\frac{1}{2}}\, \varepsilon^{\frac{N}{2}} \Big(O(1)\, \|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} + o(1)\, \|\nabla k_\varepsilon\|_{L^2(\Omega)} \Big) = o(\varepsilon^N).
\end{aligned}$$

This completes the proof of Lemma 3.24.

4.18. **Proof of Lemma 3.26.** By change of scale from (3.76), we get

$$\begin{aligned}
\varepsilon^N J_2 &= \varepsilon^N \int_{\mathbb{R}^N \setminus (\Omega/\varepsilon)} \gamma S_{U_0}(\nabla H).V_0 + \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla H_\varepsilon).V_0 \\
&\quad + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} [DT(U_0) V_0.\nabla H_\varepsilon - T(U_0).\nabla K_\varepsilon].
\end{aligned}$$

In view of (3.2) and $H \in \mathcal{V}(\mathbb{R}^N)$, the first integral on the right-hand side is the remainder of a converging integral. Thus

$$\int_{\mathbb{R}^N \setminus (\Omega/\varepsilon)} \gamma S_{U_0}(\nabla H).V_0 \;=\; o(1).$$

It follows

$$\begin{aligned}
&\varepsilon^N J_2 - o(\varepsilon^N) = \\
&\qquad \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla H_\varepsilon).V_0 + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} [DT(U_0) V_0.\nabla H_\varepsilon - T(U_0).\nabla K_\varepsilon]. \qquad (4.74)
\end{aligned}$$

Therefore gathering (3.75) and (4.74) yields

$$\begin{aligned}
\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 - o(\varepsilon^N) &:= \int_\Omega \gamma_\varepsilon \left[S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon) \right].V_0 \\
&\quad + (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} DT(U_0) V_0.(\nabla h_\varepsilon - \nabla H_\varepsilon) \qquad (4.75) \\
&\quad - (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} T(U_0).(\nabla k_\varepsilon - \nabla K_\varepsilon). \qquad (4.76)
\end{aligned}$$

Regarding the term (4.75), Hölder's inequality and estimate (3.36) imply

$$\begin{aligned}\int_{\omega_\varepsilon} DT(U_0)V_0.(\nabla h_\varepsilon - \nabla H_\varepsilon) &\leq |DT(U_0)V_0|\, |\omega|^{\frac{1}{q}}\, \varepsilon^{\frac{N}{q}}\, \|\nabla h_\varepsilon - \nabla H_\varepsilon\|_{L^p(\Omega)} \\ &= O(\varepsilon^{\frac{N}{q}})\ o(\varepsilon^{\frac{N}{p}}) = o(\varepsilon^N).\end{aligned}$$

Similarly for(4.76), Cauchy-Schwarz's inequality and estimate (3.58) entail

$$\begin{aligned}\int_{\omega_\varepsilon} T(U_0).(\nabla k_\varepsilon - \nabla K_\varepsilon) &\leq |T(U_0)|\, |\omega|^{\frac{1}{2}}\, \varepsilon^{\frac{N}{2}}\, \|\nabla k_\varepsilon - \nabla K_\varepsilon\|_{L^2(\Omega)} \\ &= O(\varepsilon^{\frac{N}{2}})\ o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N).\end{aligned}$$

It follows

$$\tilde{j}_2(\varepsilon) - \varepsilon^N J_2 \ := \ \int_\Omega \gamma_\varepsilon \left[S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon)\right].V_0 \ + \ o(\varepsilon^N). \tag{4.77}$$

Condition (7) provides

$$\begin{aligned}&\int_\Omega |S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon)| \\ &\qquad \leq \int_\Omega |\nabla h_\varepsilon - \nabla H_\varepsilon|\, (|\nabla h_\varepsilon| + |\nabla H_\varepsilon|) \left[c_0 + c_{p-3}\, (|\nabla h_\varepsilon| + |\nabla H_\varepsilon|)^{p-3}\right],\end{aligned}$$

with $c_{p-3} = 0$ if $p \in [2,3]$. Hence it follows from estimates (3.41) and (3.42) that

$$\int_\Omega |S_{U_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla H_\varepsilon)| \ = \ o(\varepsilon^N).$$

Using (4.77) completes the proof of Lemma 3.26.

4.19. **Proof of Lemma 3.27.**

(1) We first prove estimate (3.79). Since ∇v_0 is $\tilde{\beta}$-Hölder continuous in a neighborhood of $x_0 = 0$ for some $\tilde{\beta} > 0$, there exist $\delta > 0$ and $L > 0$ such that

$$|\nabla v_0(x) - V_0| \leq L\, |x|^{\tilde{\beta}}, \quad \forall x \in B(0,\delta).$$

To apply estimate (3.37), we choose $\alpha := \delta$ and $r := 1/2$. Hence for all $\varepsilon \in (0,1)$, according to estimates (3.24) and (3.37) it follows

$$\begin{aligned}&\int_\Omega |\nabla v_0 - V_0| \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) \\ &\leq \int_{B(0,\alpha\varepsilon^r)} L\, |x|^{\tilde{\beta}} \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) + 2\, \|\nabla v_0\|_{L^\infty(\Omega)} \int_{\Omega\setminus B(0,\alpha\varepsilon^r)} \left(|\nabla h_\varepsilon|^p + |\nabla h_\varepsilon|^2\right) \\ &\qquad \leq L\alpha^{\tilde{\beta}} \varepsilon^{r\tilde{\beta}}\ O(\varepsilon^N) + o(\varepsilon^N) = o(\varepsilon^N),\end{aligned}$$

which proves (3.79).

(2) For all $p \in (3,\infty)$ and for all $\lambda \in \mathbb{R}_+$ it holds $\lambda^{p-1} \leq \lambda^2 + \lambda^p$. Hence (3.80) follows immediately from estimate (3.79).

4.20. **Proof of Lemma 3.28.** Calculating the difference between (3.67) and (3.75) yields

$$\begin{aligned}j_2(\varepsilon) - \tilde{j}_2(\varepsilon) \ = \ &\int_\Omega \gamma_\varepsilon \left[S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right] \\ &+ (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \left[DT(\nabla u_0)\nabla v_0.\nabla \tilde{u}_\varepsilon - DT(U_0)V_0.\nabla h_\varepsilon\right] \\ &- (\gamma_1 - \gamma_0) \int_{\omega_\varepsilon} \left[T(\nabla u_0).\nabla \tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon\right].\end{aligned} \tag{4.78}$$

Let $\delta > 0$. Due to the continuity of ∇u_0 and ∇v_0 at point $x_0 = 0$ and to the continuity of DT, for $\varepsilon > 0$ small enough it holds

$$\max(|DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0|\,, |T(\nabla u_0) - T(U_0)|) \leq \delta \ \text{ in } \omega_\varepsilon.$$

Hence after Cauchy-Schwarz's inequality and estimates (3.43) and (3.24)

$$\begin{aligned}
&\int_{\omega_\varepsilon} |DT(\nabla u_0)\nabla v_0.\nabla \tilde{u}_\varepsilon - DT(U_0)V_0.\nabla h_\varepsilon| \\
&\qquad \leq \int_{\omega_\varepsilon} |DT(\nabla u_0)\nabla v_0|\,|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon| + \int_{\omega_\varepsilon} |DT(\nabla u_0)\nabla v_0 - DT(U_0)V_0|\,|\nabla h_\varepsilon| \\
&\qquad \leq |\omega|^{\frac{1}{2}}\,\varepsilon^{\frac{N}{2}}\left[\|DT(\nabla u_0)\nabla v_0\|_{L^\infty(\omega_\varepsilon)}\|\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon\|_{L^2(\Omega)} + \delta\,\|\nabla h_\varepsilon\|_{L^2(\Omega)}\right] \\
&\qquad\qquad \leq O(\varepsilon^{\frac{N}{2}})\ o(\varepsilon^{\frac{N}{2}}) + \ O(\varepsilon^{\frac{N}{2}})\ \delta\ O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N).
\end{aligned}$$

Similarly after Cauchy-Schwarz's inequality and estimates (3.60) and (3.53)

$$\begin{aligned}
&\int_{\omega_\varepsilon} |T(\nabla u_0).\nabla \tilde{v}_\varepsilon - T(U_0).\nabla k_\varepsilon| \\
&\qquad \leq \int_{\omega_\varepsilon} |T(\nabla u_0)|\,|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon| + \int_{\omega_\varepsilon} |T(\nabla u_0) - T(U_0)|\,|\nabla k_\varepsilon| \\
&\qquad \leq |\omega|^{\frac{1}{2}}\,\varepsilon^{\frac{N}{2}}\left[\|T(\nabla u_0)\|_{L^\infty(\Omega)}\|\nabla \tilde{v}_\varepsilon - \nabla k_\varepsilon\|_{L^2(\Omega)} + \delta\,\|\nabla k_\varepsilon\|_{L^2(\Omega)}\right] \\
&\qquad\qquad \leq O(\varepsilon^{\frac{N}{2}})\ o(\varepsilon^{\frac{N}{2}}) + \ O(\varepsilon^{\frac{N}{2}})\ \delta\ O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^N).
\end{aligned}$$

Thus (4.78) yields

$$j_2(\varepsilon) - \tilde{j}_2(\varepsilon) - o(\varepsilon^N) \ = \ \int_\Omega \gamma_\varepsilon\left[S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right], \tag{4.79}$$

which we split into three terms as

$$\begin{aligned}
&\int_\Omega \gamma_\varepsilon\left[S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon).\nabla v_0 - S_{U_0}(\nabla h_\varepsilon).V_0\right] \\
&\quad = \int_\Omega \gamma_\varepsilon\left[S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)\right].\nabla v_0 + \int_\Omega \gamma_\varepsilon\left[S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon)\right].\nabla v_0 \\
&\qquad\qquad + \int_\Omega \gamma_\varepsilon S_{U_0}(\nabla h_\varepsilon).\,(\nabla v_0 - V_0)\,. \quad (4.80)
\end{aligned}$$

(1) Regarding the first term on the right-hand side of (4.80), as $\nabla u_0 \in L^\infty(\Omega)$, it follows from condition (7) that

$$\begin{aligned}
&\int_\Omega |S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)| \\
&\qquad \leq \int_\Omega |\nabla \tilde{u}_\varepsilon - \nabla h_\varepsilon|\,(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|)\left[c_0 + c_{p-3}\,(|\nabla \tilde{u}_\varepsilon| + |\nabla h_\varepsilon|)^{p-3}\right]
\end{aligned}$$

with $c_{p-3} = 0$ for all $p \in [2, 3]$. Thus estimates (3.44) and (3.45) entail

$$\int_\Omega |S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)| \ = \ o(\varepsilon^N).$$

As $\nabla v_0 \in L^\infty(\Omega)$, it follows

$$\int_\Omega \gamma_\varepsilon\left[S_{\nabla u_0}(\nabla \tilde{u}_\varepsilon) - S_{\nabla u_0}(\nabla h_\varepsilon)\right].\nabla v_0 = o(\varepsilon^N).$$

(2) Regarding the second term on the right-hand side of (4.80), as $\nabla u_0 \in L^\infty(\Omega)$, according to condition (8)

$$\int_\Omega |S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon)| \quad \leq \quad \int_\Omega |\nabla u_0 - U_0| \left[d_0 \, |\nabla h_\varepsilon|^2 + d_{p-4} \, |\nabla h_\varepsilon|^{p-2} \right]$$

with $d_{p-4} = 0$ for all $p \in [2,4]$. Thus estimates (3.38) and (3.39) entail

$$\int_\Omega |S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon)| \quad = \quad o(\varepsilon^N).$$

As $\nabla v_0 \in L^\infty(\Omega)$, it follows

$$\int_\Omega \gamma_\varepsilon \left[S_{\nabla u_0}(\nabla h_\varepsilon) - S_{U_0}(\nabla h_\varepsilon) \right] . \nabla v_0 = o(\varepsilon^N).$$

(3) Regarding the third term on the right-hand side of (4.80), according to (3.2), it holds

$$\int_\Omega |S_{U_0}(\nabla h_\varepsilon)| \; |\nabla v_0 - V_0| \quad \leq \quad \int_\Omega |\nabla v_0 - V_0| \left[c_0 \, |\nabla h_\varepsilon|^2 + c_{p-3} \, |\nabla h_\varepsilon|^{p-1} \right]$$

with $c_{p-3} = 0$ for all $p \in [2,3]$. Hence it follows from estimates (3.79) and (3.80) that

$$\int_\Omega |S_{U_0}(\nabla h_\varepsilon)| \; |\nabla u_0 - U_0| \quad = \quad o(\varepsilon^N).$$

Gathering the above estimates, (4.80) and (4.79) completes the proof of Lemma 3.28.

5. Conclusion

In this article, we first analyzed specific issues arising in the process of obtaining a topological asymptotic expansion for a second order quasilinear elliptic equation, by comparison with a linear elliptic equation. When trying to define the variation of the direct state at scale 1 in $\mathbb{R}^N$, it turns out that this variation can be defined by applying the Minty-Browder theorem to a specific nonlinear operator, which is derived from the considered quasilinear equation. The requirements of the Minty-Browder theorem bring into light a two-norms discrepancy involving the L^p and the L^2 norms of the gradient. They require to consider at the same time

- a functional space which is equipped with a norm giving control on both the L^p and the L^2 norms of the gradient and which enjoys in addition a Poincaré inequality;
- a quasilinear elliptic equation whose operator enjoys both p- and 2- ellipticity properties.

The first condition justifies that we built the quotient weighted Sobolev space $\mathcal{V}(\mathbb{R}^n)$ and the quotient weighted Hilbert space $\mathcal{H}(\mathbb{R}^N)$ in appendix A. The second condition explains why in section 3 we restricted ourselves to a specific class of quasilinear equations.

Several other key features of the linear method had to be adapted to the nonlinear case. In particular, implementing the method required to:

(1) ensure duality between the variation of the direct state and the corresponding variation of the adjoint state at each stage of approximation;
(2) determine the spatial decay of the variation of the direct state at scale 1;
(3) determine with respect to the variation of the direct state, what does mean 'far away from the perturbation' by opposition to 'close to the perturbation'.

As a result, our main contribution is Theorem 3.5 which provides the topological asymptotic expansion for quasilinear elliptic equations of the considered class.

Our belief is that the doorway of topological asymptotic expansions for quasilinear elliptic equations is now opened. As topological asymptotic expansions will gradually become available for larger classes of nonlinear equations and of functionals, the scope of attainable applicative tasks should significantly broaden, in particular in shape optimization and in imaging. Further research can be pursued in several directions, like for instance to obtain similar topological asymptotic expansions for larger classes of quasilinear elliptic equations, including degenerate equations such as the p-Laplace equation, or to tackle models of non-linear elasticity.

Appendix A. Weighted and quotient Sobolev spaces

The purpose of this appendix is to build an appropriate reflexive Banach space so as to define the variation of the direct state at scale 1 in $\mathbb{R}^N$. In such a space, the variational form defining this variation has to comply with the requirements of the Minty-Browder theorem. The main result of this section is Proposition A.5, which ensures the required coercivity property involving both the L^p and the L^2 norms of the gradient. Similarly, we build an appropriate Hilbert space so as to define the variation of the adjoint state at scale 1 in $\mathbb{R}^N$.

The building scheme of such spaces is classical. We follow the approach of [9], which directly provides Poincaré inequalities in the preliminary space $\mathcal{W}(\mathbb{R}^N)$ as well as in the Hilbert space $\mathcal{H}(\mathbb{R}^N)$. We shall take one more step to obtain a similar result in our main working space $\mathcal{V}(\mathbb{R}^N)$.

A.1. **Weighted Sobolev spaces.** We define the weight function $w_p : \mathbb{R}^N \to \mathbb{R}$ as follows: for all $x \in \mathbb{R}^N$,

$$w_p(x) := \begin{cases} \Big(1+|x|^2\Big)^{-\frac{1}{2}} & \text{if } p \neq N, \\ \Big(1+|x|^2\Big)^{-\frac{1}{2}} \Big(\log(2+|x|^2)\Big)^{-1} & \text{if } p = N. \end{cases}$$

For all open subset $\mathcal{O} \subset \mathbb{R}^N$, recall we denote $\mathcal{D}'(\mathcal{O})$ the space of distributions in $\mathcal{O}$. Let the space

$$\mathcal{V}^w(\mathcal{O}) := \big\{u \in \mathcal{D}'(\mathcal{O}) \ ; \ w_p u \in L^p(\mathcal{O}), \nabla u \in L^p(\mathcal{O}) \cap L^2(\mathcal{O})\big\}$$

endowed with the norm defined by

$$\|u\|_{\mathcal{V}^w(\mathcal{O})} \quad := \quad \|w_p u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^2(\mathcal{O})}\,, \quad \forall u \in \mathcal{V}^w(\mathcal{O}).$$

For technical purposes it is useful to define the larger space

$$\mathcal{W}^w(\mathcal{O}) := \big\{u \in \mathcal{D}'(\mathcal{O}) \ ; \ w_p u \in L^p(\mathcal{O}), \nabla u \in L^p(\mathcal{O})\big\}$$

endowed with the norm defined by

$$\|u\|_{\mathcal{W}^w(\mathcal{O})} \quad := \quad \|w_p u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^p(\mathcal{O})}\,, \quad \forall u \in \mathcal{W}^w(\mathcal{O}).$$

Then we define the space

$$\mathcal{H}^w(\mathcal{O}) := \big\{u \in \mathcal{D}'(\mathcal{O}) \ ; \ w_2 u \in L^2(\mathcal{O}), \nabla u \in L^2(\mathcal{O})\big\}$$

endowed the inner product defined by

$$\langle u, v\rangle_{\mathcal{H}^w(\mathcal{O})} := \langle w_2 u, w_2 v\rangle_{L^2(\mathcal{O})} + \langle \nabla u, \nabla v\rangle_{L^2(\mathcal{O})}\,, \quad \forall u, v \in \mathcal{H}^w(\mathcal{O}).$$

Obviously, the three normed spaces $\mathcal{V}^w(\mathcal{O})$, $\mathcal{W}^w(\mathcal{O})$ and $\mathcal{H}^w(\mathcal{O})$ coincide when $p = 2$.

The following Lemma A.1 can be proved by standard arguments, see e.g. [1, 28].

Lemma A.1. *The spaces $\mathcal{W}^w(\mathcal{O})$ and $\mathcal{V}^w(\mathcal{O})$ endowed with the norms $\|.\|_{\mathcal{W}^w(\mathcal{O})}$ and $\|.\|_{\mathcal{V}^w(\mathcal{O})}$, respectively, are reflexive separable Banach spaces. The space $\mathcal{H}^w(\mathcal{O})$ endowed with the inner product $\langle .,.\rangle_{\mathcal{H}^w(\mathcal{O})}$ is a separable Hilbert space.*

A.2. **Quotient weighted Sobolev spaces.** It is straightforwardly checked that the constant functions belong to $\mathcal{W}^w(\mathbb{R}^N)$ if and only if $p > N$. Therefore we set

$$P_p = \begin{cases} \{0\} & \text{if} \quad p \leq N, \\ \mathbb{R} & \text{if} \quad p > N, \end{cases}$$

and define the quotient space

$$\mathcal{W}(\mathbb{R}^N) := \mathcal{W}^w(\mathbb{R}^N)/P_p.$$

Note that, should we have considered a Sobolev space $\mathcal{W}^w(\mathbb{R}^N)$ of higher order, the set P_p would have contained higher order polynomials, see [9, 17]. The space $\mathcal{W}(\mathbb{R}^N)$ is equipped with its natural quotient norm

$$\|[u]\|_{\mathcal{W}(\mathbb{R}^N)} := \inf_{m \in P_p} \|u + m\|_{\mathcal{W}^w(\mathbb{R}^N)}, \quad \forall\ [u] \in \mathcal{W}(\mathbb{R}^N) \tag{A.1}$$

where $u \in \mathcal{W}^w(\mathbb{R}^N)$ is an arbitrary representative of the class $[u]$.

Since $\mathcal{W}^w(\mathbb{R}^N)$ is a reflexive Banach space and P_p is a closed subspace then $\mathcal{W}(\mathbb{R}^N)$ is still a reflexive Banach space (see e.g. [28], chapter XI, §11.2).

Similarly constant functions belong to $\mathcal{V}^w(\mathbb{R}^N)$ if and only if they belong to $\mathcal{W}^w(\mathbb{R}^N)$. Thus we define likewise

$$\mathcal{V}(\mathbb{R}^N) := \mathcal{V}^w(\mathbb{R}^N)/P_p,$$

equipped with the norm

$$\|[u]\|_{\mathcal{V}(\mathbb{R}^N)} := \inf_{m \in P_p} \|u + m\|_{\mathcal{V}^w(\mathbb{R}^N)}, \quad \forall\ [u] \in \mathcal{V}(\mathbb{R}^N). \tag{A.2}$$

In a similar way we construct the Hilbert space

$$\mathcal{H}(\mathbb{R}^N) := \mathcal{H}^w(\mathbb{R}^N)/P_2.$$

A.3. **Poincaré inequality in $\mathcal{W}(\mathbb{R}^N)$.** The following key result is proven in [9].

Theorem A.2. *There exists $c > 0$ such that*

$$\|[u]\|_{\mathcal{W}(\mathbb{R}^N)} \leq c \|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \forall\ [u] \in \mathcal{W}(\mathbb{R}^N),$$

where $u \in \mathcal{W}^w(\mathbb{R}^N)$ is any representative of the class $[u]$.

For all $[u] \in \mathcal{W}(\mathbb{R}^N)$, let $u \in \mathcal{W}^w(\mathbb{R}^N)$ be an arbitrary element in the class $[u]$. Endow $\mathcal{W}(\mathbb{R}^N)$ with the semi-norm

$$|[u]|_{\mathcal{W}(\mathbb{R}^N)} := \|\nabla u\|_{L^p(\mathbb{R}^N)}. \tag{A.3}$$

Theorem A.2 can be rephrased as follows.

Corollary A.3. *The semi-norm $|\ .\ |_{\mathcal{W}(\mathbb{R}^N)}$ and the norm $\|\ .\ \|_{\mathcal{W}(\mathbb{R}^N)}$ are equivalent in $\mathcal{W}(\mathbb{R}^N)$.*

A.4. **Poincaré inequality and coercivity in $\mathcal{V}(\mathbb{R}^N)$.** Let $[u] \in \mathcal{V}(\mathbb{R}^N)$ and $u \in \mathcal{V}^w(\mathbb{R}^N)$ be any element of the class $[u]$. Endow $\mathcal{V}(\mathbb{R}^N)$ with the semi-norm given by

$$|[u]|_{\mathcal{V}(\mathbb{R}^N)} := \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)}. \tag{A.4}$$

Theorem A.2 also implies the following, whose straightforward proof is left to the reader.

Corollary A.4. *The semi-norm $|\ .\ |_{\mathcal{V}(\mathbb{R}^N)}$ and the norm $\|\ .\ \|_{\mathcal{V}(\mathbb{R}^N)}$ are equivalent in $\mathcal{V}(\mathbb{R}^N)$.*

We can now state the main result of this appendix, which enables to prove the combined p- and 2- coercivity property.

Proposition A.5. *For all $[u] \in \mathcal{V}(\mathbb{R}^N)$, denote by $u \in \mathcal{V}^w(\mathbb{R}^N)$ any element in the class $[u]$. Then it holds*

$$\lim_{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)} \to \infty} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|[u]\|_{\mathcal{V}(\mathbb{R}^N)}} = +\infty.$$

Proof. To study the limit at infinity, given the equivalence stated in Corollary A.4, one can assume that
$$|[u]|_{\mathcal{V}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^2(\mathbb{R}^N)} \geq 1.$$

(1) If $\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq 1$, then it holds
$$\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \geq \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \geq \frac{(|[u]|_{\mathcal{V}(\mathbb{R}^N)} - 1)^2}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}}.$$

(2) If $\|\nabla u\|_{L^p(\mathbb{R}^N)} > 1$, since $2 \leq p < \infty$, then it holds
$$\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \geq \frac{\|\nabla u\|^2_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \geq \frac{1}{2}\,|[u]|_{\mathcal{V}(\mathbb{R}^N)}\,.$$

Thus we have
$$\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \quad \geq \quad \min\left(\frac{(|[u]|_{\mathcal{V}(\mathbb{R}^N)} - 1)^2}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}}, \frac{|[u]|_{\mathcal{V}(\mathbb{R}^N)}}{2}\right).$$

Hence
$$\lim_{|[u]|_{\mathcal{V}(\mathbb{R}^N)}\to\infty} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{|[u]|_{\mathcal{V}(\mathbb{R}^N)}} \;=\; +\infty.$$

In view of the equivalence stated in Corollary A.4, we obtain the claimed limit. □

A.5. **Poincaré inequality and coercivity in $\mathcal{H}(\mathbb{R}^N)$.** For all $[u] \in \mathcal{H}(\mathbb{R}^N)$, denote by $u \in \mathcal{H}^w(\mathbb{R}^N)$ any element in the class $[u]$. Endow $\mathcal{H}(\mathbb{R}^N)$ with the semi-norm
$$|[u]|_{\mathcal{H}(\mathbb{R}^N)} := \|\nabla u\|_{L^2(\mathbb{R}^N)}\,.$$

Applying Corollary A.4 in the case $p = 2$ straightforwardly yields:

Corollary A.6. *The semi-norm $|\ .\ |_{\mathcal{H}(\mathbb{R}^N)}$ and the norm $\|\ .\ \|_{\mathcal{H}(\mathbb{R}^N)}$ are equivalent in $\mathcal{H}(\mathbb{R}^N)$.*

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